Math 846

Lecture 20

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter $D \geq 1$. Assume Γ is Q-polynomial wrt $\{E_i\}_{i=0}^{D}$. Fix $x \in X$ and write $T = T(x)$ etc.

Lec 20
1

LEM 3 For $0 \leq i, j, r \leq D$,

(i) $$E_i^* A^r E_j^* = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

(ii) $$E_i (A^*)^r E_j = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

pf (i) Recall

$$E_i^* A_r E_j^* = 0 \quad \text{iff} \quad p_{ij}^r = 0$$

A_r = polynomial in A with degree exactly r

$$p_{ij}^r = \begin{cases} 0 & \text{if } r < |i-j| \\ \neq 0 & \text{if } r = |i-j| \end{cases}$$

Result follows.

(ii) Similar

$\square$

It turns out, Thm 1, Thm 2 follow from Lem 3 alone. To prove Thm 1, Thm 2 we will give a sequence of results.

Each result has a "dual" obtained by interchanging $A \leftrightarrow A^*$ and $E_i \leftrightarrow E_i^*$ for $0 \le i \le D$.

We will not explicitly state each dual result.

LEM 4 For $0 \le i, j, r, s \le D$

$$E_i^* A^r A^* A^s E_j^* = \begin{cases} \theta^*_{j+s} & E_i^* A^{r+s} E_j^* \quad \text{if } i-j = r+s \\ \theta^*_{j-s} & E_i^* A^{r+s} E_j^* \quad \text{if } j-i = r+s \\ 0 & \quad \text{if } |i-j| > r+s \end{cases}$$

pf Observe

$$E_i^* A^r A^* A^s E_j^* = E_i^* A^r \left(\sum_{h=0}^{D} \theta_h^* E_h^* \right) A^s E_j^*$$

and use Lem 3 (i). $\square$

LEM 5 We have

$$\text{Span}\{ RA^*S - SA^*R \mid R, S \in M \}$$
$$= \{ YA^* - A^*Y \mid Y \in M \}$$

pf Abbrev

$$L_i = E_0 + E_1 + \cdots + E_i \qquad (0 \le i \le D)$$

and

$$E_{-1} = 0, \qquad E_{D+1} = 0$$

Claim: For $0 \le i \le D$,

$$E_i A^* E_{i+1} - E_{i+1} A^* E_i = L_i A^* - A^* L_i$$

pf cl For $0 \le j \le D$

$$E_j A^* = E_j A^* (E_0 + E_1 + \cdots + E_D)$$
$$= E_j A^* (E_{j-1} + E_j + E_{j+1}) \qquad (*)$$

and similarly

$$A^* E_j = (E_{j-1} + E_j + E_{j+1}) A^* E_j \qquad (**)$$

Now sum each of $(*)$, $(**)$ over $j = 0, 1, \ldots, i$ and take the difference to get the claim.

Recall $\{E_i\}_{i=0}^{D}$ is a basis for M, so

$\{L_i\}_{i=0}^{D}$ is a basis for M.

Observe

$$\mathrm{Span}\{RA^*S - SA^*R \mid R, S \in M\}$$

$$= \mathrm{Span}\{E_iA^*E_j - E_jA^*E_i \mid 0 \le i,j \le D\}$$

$$= \mathrm{Span}\{E_iA^*E_{i+1} - E_{i+1}A^*E_i \mid 0 \le i \le D\}$$

$$= \mathrm{Span}\{L_iA^* - A^*L_i \mid 0 \le i \le D\}$$

$$= \{YA^* - A^*Y \mid Y \in M\}$$

$\square$

Notation 6

(i) Given $\beta, \gamma, \delta \in \mathbb{F}$ define a 2-variable polynomial

$$p(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma(\lambda+\mu) - \delta$$

(ii) Given $\beta, \gamma^*, \delta^* \in \mathbb{F}$ define a 2-variable polynomial

$$p^*(\lambda,\mu) = \lambda^2 - \beta\lambda\mu + \mu^2 - \gamma^*(\lambda+\mu) - \delta^*$$

LEM 7 For $\beta, \gamma, \delta \in \mathbb{F}$

$$0 = \left[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^* \right] \qquad (*)$$

iff $\qquad (**)$

$$P(\theta_{i-1}, \theta_i) = 0 \quad \text{for } 1 \le i \le D.$$

pf let C = RHS of $(*)$.

$$C = (E_0 + E_1 + \cdots + E_D)\, C\, (E_0 + E_1 + \cdots + E_D)$$
$$= \sum_{i=0}^{D} \sum_{j=0}^{D} E_i C E_j$$

For $0 \le i, j \le D$ use $E_i A = \theta_i E_i$ and $AE_j = \theta_j E_j$

to get

$$E_i C E_j = E_i A^* E_j\, (\theta_i - \theta_j)\, P(\theta_i, \theta_j)$$

$(*) \Rightarrow (**)$ For $1 \le i \le D$ show $P(\theta_{i-1}, \theta_i) = 0$

$C = 0$ so

$$0 = E_{i-1} C E_i$$
$$= \underbrace{E_{i-1} A^* E_i}_{\ne 0}\ \underbrace{(\theta_{i-1} - \theta_i)}_{\ne 0}\ P(\theta_{i-1}, \theta_i)$$

so $P(\theta_{i-1}, \theta_i) = 0$

$(**) \Rightarrow (*)$: $\quad P$ is symmetric in its arguments

so
$$P(\theta_i, \theta_{i-1}) = 0 \qquad 1 \le i \le D$$

To show $C = 0$, show
$$E_i C E_j = 0 \qquad (0 \le i, j \le D)$$

For $0 \le i, j \le D$

If $|i-j| > 1$ then $E_i A^* E_j = 0$ so $E_i C E_j = 0$

If $|i-j| = 1$ then $P(\theta_i, \theta_j) = 0$ so $E_i C E_j = 0$

If $i = j$ then $\theta_i - \theta_j = 0$ so $E_i C E_j = 0$

In each case $E_i C E_j = 0$ so $C = 0$ $\qquad \square$

Notation

Let $d \in \mathbb{N}$ and $\sigma_i \in \mathbb{F}$ for $0 \le i \le d$

Def 8 Let $\beta, \gamma, \delta \in \mathbb{F}$.

(i) The sequence $\{\sigma_i\}_{i=0}^{d}$ is <u>recurrent</u> whenever

$\sigma_{i-1} \ne \sigma_i$ for $1 \le i \le d$ and

$$\frac{\sigma_{i-2} - \sigma_{i+1}}{\sigma_{i-1} - \sigma_i}$$

is indep of i for $2 \le i \le d-1$

(ii) The sequence $\{\sigma_i\}_{i=0}^{d}$ is <u>β-recurrent</u> whenever

$$\sigma_{i-2} - (\beta+1)\sigma_{i-1} + (\beta+1)\sigma_i - \sigma_{i+1} = 0$$

for $2 \le i \le d-1$

(iii) The sequence $\{\sigma_i\}_{i=0}^{d}$ is <u>(β, γ)-recurrent</u> whenever

$$\sigma_{i-1} - \beta\sigma_i + \sigma_{i+1} = \gamma$$

for $1 \le i \le d-1$

(iv) The sequence $\{\sigma_i\}_{i=0}^{d}$ is (β, γ, δ)-recurrent whenever

$$\sigma_{i-1}^2 - \beta\sigma_{i-1}\sigma_i + \sigma_i^2 - \gamma(\sigma_{i-1} + \sigma_i) = \delta$$

for $1 \le i \le d$

LEM 9 TFAE:

(i) $\{\sigma_i\}_{i=0}^{d}$ is recurrent

(ii) $\sigma_{i-1} \neq \sigma_i$ for $2 \le i \le d-1$, and $\exists \beta \in \mathbb{F}$ st $\{\sigma_i\}_{i=0}^{d}$ is β-recurrent

Suppose (i), (ii) and $d \geq 3$. Then the common value of

$$\frac{\sigma_{i-2} - \sigma_{i+1}}{\sigma_{i-1} - \sigma_i}$$

is equal to $\beta+1$.

pf Routine

LEM 10 For $\beta \in \mathbb{F}$ TFAE:

(i) $\{\sigma_i\}_{i=0}^{d}$ is β-recurrent

(ii) $\exists \gamma \in \mathbb{F}$ st $\{\sigma_i\}_{i=0}^{d}$ is (β, γ)-recurrent

pf Routine

LEM 11 The following (i), (ii) hold for $\beta, \gamma \in \mathbb{F}$

(i) Assume $\{\sigma_i\}_{i=0}^{d}$ is (β, γ)-recurrent. Then $\exists\, \delta \in \mathbb{F}$ st $\{\sigma_i\}_{i=0}^{d}$ is (β, γ, δ)-recurrent.

(ii) Assume $\{\sigma_i\}_{i=0}^{d}$ is (β, γ, δ)-recurrent, and $\sigma_{i-1} \neq \sigma_{i+1}$ for $1 \le i \le d-1$, then $\{\sigma_i\}_{i=0}^{d}$ is (β, γ)-recurrent.

pf Define

$$p_i = \sigma_{i-1}^2 - \beta\, \sigma_{i-1}\sigma_i + \sigma_i^2 - \gamma\,(\sigma_{i-1} + \sigma_i) \qquad 1 \le i \le d$$

For $1 \le i \le d-1$

$$p_i - p_{i+1} = (\sigma_{i-1} - \sigma_{i+1})(\sigma_{i-1} - \beta\,\sigma_i + \sigma_{i+1} - \gamma)$$

Result follows $\square$