

Combinatorics 475

Lecture 1

Ch 2 Permutations and combinations

2.1 4 basic counting principles

Given a set S

A partition of S is a decomposition

of S into subsets such that

each element of S is contained in exactly one of the subsets.

Ex $S = \{1, 2, 3, a, b\}$

$$A = \{1, a\}$$

$$B = \{2, b\}$$

$$C = \{3\}$$

A, B, C partition S

For a set S define

$$|S| = \text{number of elements in } S$$

"cardinality of S "

"size of S "

For above example

$$\begin{aligned} |S| &= 5 = 2 + 2 + 1 \\ &= |A| + |B| + |C| \end{aligned}$$

This illustrates

Addition Principle

Suppose a set S is partitioned

into subsets $S_1, S_2, \dots, S_k$.

Then

$$|S| = |S_1| + |S_2| + \dots + |S_k|$$

Given sets A, B

their Cartesian product denoted $A \times B$

is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$

Ex $A = \{1, 2, 4\}$ $B = \{r, s\}$

els of $A \times B$ are in table:

	B	
	r	s
A	1	(1, r) (1, s)
	2	(2, r) (2, s)
	4	(4, r) (4, s)

Obs

$$|A \times B| = 6 = 3 \times 2 = |A| |B|$$

In general for any finite sets A, B

$$|A \times B| = |A| |B|$$

Ex For an integer $n \geq 1$ Find the number of sequences

$$a_1 a_2 \dots a_n$$

$$a_i \in \{0, 1\} \quad 1 \leq i \leq n$$

Sol Def $A = \{0, 1\}$

View set of sequences as Cartesian product

$$\underbrace{A \times A \times \dots \times A}_{n \text{ copies}}$$

$$\begin{aligned} \# \text{ sequences} &= |A|^n \\ &= 2^n \end{aligned}$$

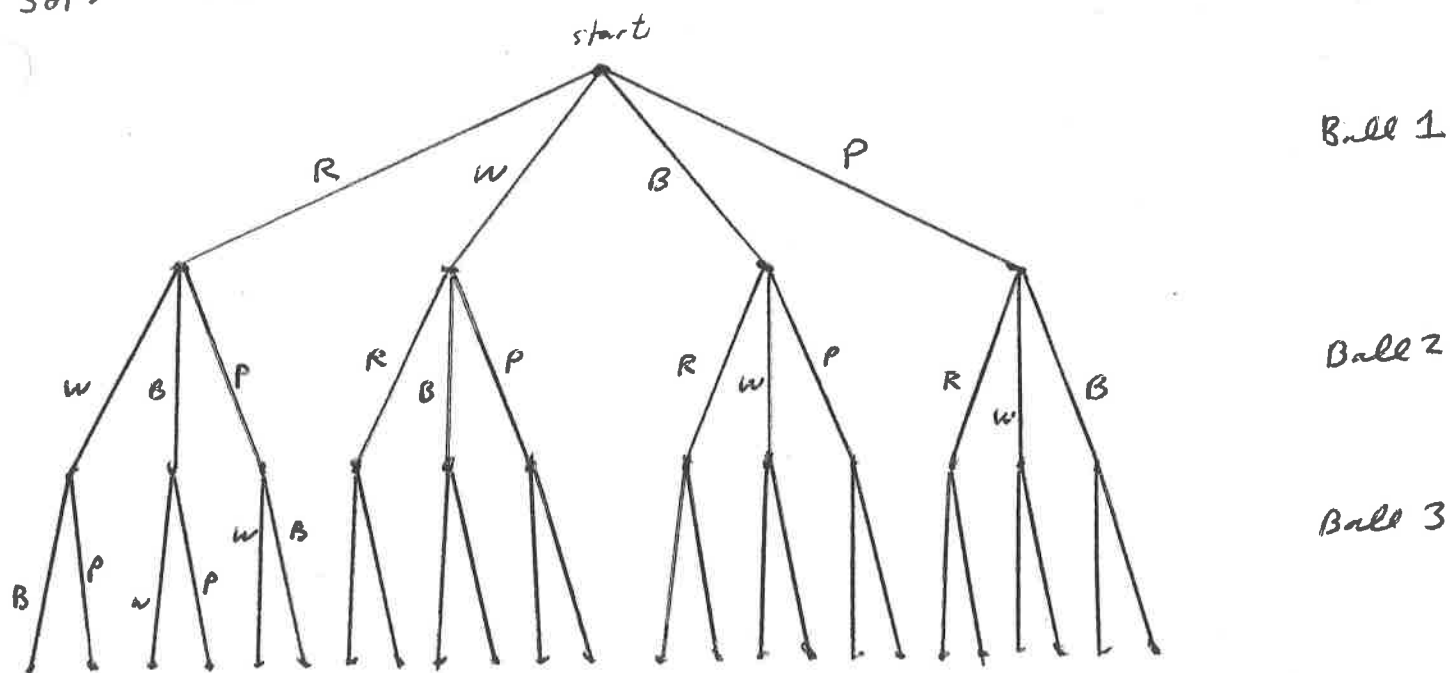
Prob: A box contains

- 1 Red ball
- 1 white ball
- 1 blue ball
- 1 purple ball

3 balls are withdrawn, one at a time, without replacement and colors noted.

How many outcomes are possible?

Sol: Illustrate with "tree diagram"



Each outcome is represented by a path in the tree diagram, from "start" to a "leaf"

$$\# \text{outcomes} = 4 \times 3 \times 2 = 24$$

Above examples illustrate the

Lec 1 6

Multiplication principle

Consider a multistage experiment with k stages

Assume:

- stage 1 has n_1 possible outcomes
- stage 2 has n_2 possible outcomes
(regardless of result of stage 1)
- stage 3 has n_3 possible outcomes
(regardless of results of stages 1, 2)
- $\vdots$
- stage k has n_k possible outcomes
(regardless of results of prev stages)

Then the total number of outcomes is

$$n_1 \times n_2 \times \dots \times n_k$$

Ex For an integer $n \geq 0$

the set $\{1, 2, \dots, n\}$ has how many subsets?

Sol. View the situation as an n -stage experiment in which we select a subset

stage	to do	# choices
1	decide if "1" in subset	2
2	decide if "2" in subset	2
n	decide if " n " in subset	2

By mult principle the # subsets is 2^n

Given sets $S \subseteq U$

The complement of S in U is

$$\bar{S} = \{x : x \in U \text{ and } x \notin S\}$$

We have partition

$$U = S \cup \bar{S} \quad (\text{disjoint})$$

So $|U| = |S| + |\bar{S}|$

In other words

Subtraction principle :

For $|U| < \infty$ and $S \subseteq U$,

$$|\bar{S}| = |U| - |S|$$

Ex

A product code consists of a string
of 4 dist letters taken from the set
 $\{A, H, Q, T, Z\}$

How many such codes contain the letter "H" ?

Sol

View

U = set of all product codes

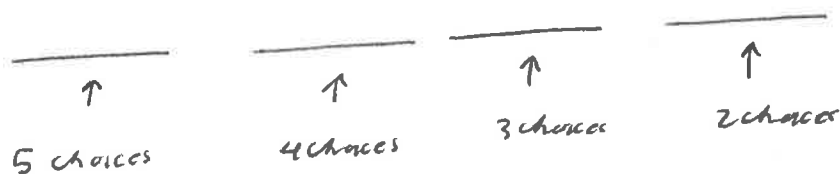
S = set of product codes that do not contain H

Find $|\bar{S}|$ using

$$|\bar{S}| = |U| - |S|$$

Find $|U|$:

Construct prod code by filling in blanks left to right

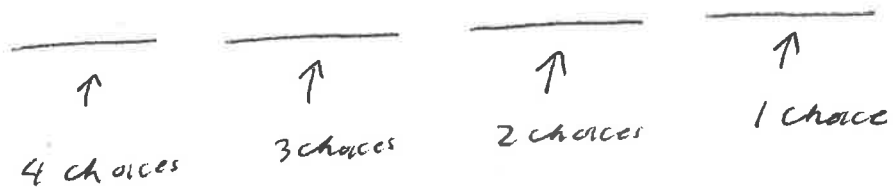


By mult principle

$$|U| = 5 \times 4 \times 3 \times 2$$

$$= 120$$

Find $|S|$:



$$|S| = 4 \times 3 \times 2 \times 1 \\ = 24$$

$$|\overline{S}| = 120 - 24 \\ = 96$$

□

Division principle

Given a finite set U partition U into subsets with equal cardinality.

Then

$$\text{the number of subsets} = \frac{|U|}{\text{cardinality of each subset}}$$

Ex Given integers $3 \leq k \leq n$

We create a pearl necklace as follows.

Given n distinct pearls

Necklace uses k of these pearls.

How many necklaces are possible?

[two necklaces are same if they differ by rot / refl]

Sol: Call the pearls $1, 2, \dots, n$

Define

$U =$ set of all sequences of k pearls taken from $\{1, 2, \dots, n\}$

First find $|U|$:

To select an element of U , fill in blanks left to right.



By multiplication principle

$$|U| = n(n-1) \dots (n-k+1)$$

Define a binary relation $\sim$ on U as follows.

$$\forall x, y \in U$$

$x \sim y$ whenever y is obtained from x by rot/refl

$\sim$ is an equiv relation.

Each equiv class consists of exactly $2k$ elements

$$\begin{aligned} & a_1 a_2 \dots a_k \\ & a_2 a_3 \dots a_k a_1 \\ & a_3 \dots a_k a_1 a_2 \\ & \vdots \\ & a_k a_1 \dots a_{k-1} \end{aligned}$$

$$\begin{aligned} & a_k a_{k-1} \dots a_1 \\ & a_1 a_k \dots a_2 \\ & a_2 a_1 a_k \dots a_3 \\ & \vdots \\ & a_{k-1} \dots a_1 a_k \end{aligned}$$

Each necklace corresponds to an equiv class of $\sim$

$$\# \text{necklaces} = \# \text{equiv class for } \sim$$

$$= \frac{|U|}{\text{card of each equiv class}}$$

$$= \frac{n(n-1) \dots (n-k+1)}{2k}$$

