Lecture 1

Spring 2021/2022

Math 846 Algebraic Graph Theory

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References

- Bannai and Ito. Algebraic Combinatorics I: Association Schemes (Benjamin Cummings 1984)

- Brouwer, Cohen, Neumaier. Distance-Regular Graphs (Springer Verlag 1989)

- Biggs. Algebraic Graph Theory 2nd Ed (Cambridge 1993)

1. Introduction

Recall the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$

and integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$

Fix a field $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Let

X = nonempty finite set $\qquad n = |X|$

Let $\mathbb{F}^X$ = vector space $\mathbb{F}^n$ (column vectors with coordinates indexed by X and entries in $\mathbb{F}$)

Abbreviate $V = \mathbb{F}^X$ "standard module"

Endow V with a Hermitean bilinear form

$$\langle u, v \rangle = u^t \bar{v} \qquad u, v \in V$$

t = transpose, $\quad \bar{v}$ = complex conjugate of v

For $u \in V$,

$$\|u\|^2 = \langle u, u \rangle \in \mathbb{R}$$

$$\|u\|^2 \geq 0$$

$$\|u\|^2 = 0 \quad \text{iff } u = 0$$

For $x \in X$ define

$$\hat{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x \text{ coord}$$

For $x, y \in X$

$$\langle \hat{x}, \hat{y} \rangle = \delta_{xy} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

So

$\{ \hat{x} \mid x \in X \}$ is an orthonormal basis for V

Write $\mathbb{1} = \sum_{x \in X} \hat{x}$ "all 1's vector"

$Mat_X(\mathbb{F})$ = set of all square matrices with rows/cols indexed by X and entries in $\mathbb{F}$

$Mat_X(\mathbb{F})$ is algebra over $\mathbb{F}$ with identity

$$I = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$$

$\mathrm{Mat}_X(\mathbb{F})$ acts on V by left multiplication.

Linear algebra reminder

Given $B \in \mathrm{Mat}_X(\mathbb{F})$, For $u, v \in V$

$$\langle Bu, v \rangle = \langle u, \overline{B}^t v \rangle$$

Call B Hermitean whenever $\overline{B}^t = B$. In this case, B is diagonalizable over $\mathbb{F}$. By the eigenvalues of B we mean the roots of the characteristic polynomial of B these eigenvalues form a multiset. Each eigenvalue is in $\mathbb{R}$

By the distinct eigenvalues of B we mean the roots of the minimal polynomial of B. These eigenvalues form the set that underlies the above multiset

Let $\{\theta_i\}_{i=0}^r$ denote an ordering of the distinct eigenvalues of B

$$\left[\text{usually } \theta_0 > \theta_1 > \theta_2 > \cdots > \theta_r \right]$$

For $0 \le i \le r$ let

$$V_i = \text{eigenspace of } B \text{ for } \theta_i$$

One checks

$$\langle V_i, V_j \rangle = 0 \quad \text{if } i \neq j \quad (0 \le i, j \le r)$$

So

$$V = \sum_{i=0}^{r} V_i \qquad \text{(orthogonal direct sum)}$$

For $0 \le i \le r$ define $E_i \in Mat_X(\mathbb{R})$ st

$$(E_i - I)V_i = 0$$

$$E_i V_j = 0 \quad \text{if } i \neq j \qquad (0 \le j \le r)$$

We have

$$E_i E_j = \delta_{ij} E_i \qquad (0 \le i, j \le r)$$

$$I = \sum_{i=0}^{r} E_i$$

$$V_i = E_i V \qquad (0 \le i \le r)$$

$$B E_i = E_i B = \theta_i E_i \qquad (0 \le i \le r)$$

$$B = \sum_{i=0}^{r} \theta_i E_i$$

$$E_i = \prod_{\substack{0 \le j \le r \\ j \neq i}} \frac{B - \theta_j I}{\theta_i - \theta_j}$$

$$\overline{E_i}^t = E_i \qquad \text{(easier)}$$

let M = subalgebra of $\mathrm{Mat}_X(\mathbb{F})$ generated by B

vector space M has a basis $\{B^i\}_{i=0}^r$

and a basis $\{E_i\}_{i=0}^r$

Also

$$0 = \prod_{i=0}^{r} (B - \theta_i I)$$

Fr easier

Call E_i the primitive idempotent of B for θ_i

Define

$$m_i = \dim V_i$$

$$(= \operatorname{rank} E_i = \operatorname{trace} E_i)$$

call m_i the multiplicity of θ_i

the spectrum of B is the array

$$\operatorname{Spec}(B) = \begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_r \\ m_0 & m_1 & \cdots & m_r \end{pmatrix}$$

A <u>graph</u> is an ordered pair $\Gamma = (X, R)$

where

X = nonempty finite set (the vertices)

R = set of distinct 2-element subsets of X.
(the edges)

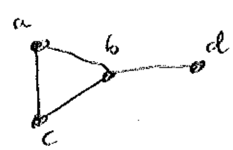
For distinct $x, y \in X$

x, y are <u>adjacent</u> whenever they form an edge

EX $X = \{a, b, c, d\}$

$R = \{ab, bc, ca, bd\}$

Γ:



connect adjacent vertices by an arc

Our graphs are

- finite
- undirected
- no loops
- no multiple edges

For a graph $\Gamma = (X, R)$

For $x \in X$ define

$\Gamma(x) =$ set of vertices adjacent x

$k(x) = |\Gamma(x)|$ "valency of x" "degree of x"

Γ is <u>regular with valency k</u> whenever $k(x) = k$ $\forall x \in X$

For an integer $\ell \geq 0$

A <u>walk of length ℓ</u> in Γ is a sequence of vertices

$x_0, x_1, \ldots, x_\ell$

st x_{i-1}, x_i are adjacent for $1 \leq i \leq \ell$

A <u>path of length ℓ</u> in Γ is a walk

$x_0, x_1, \ldots, x_\ell$

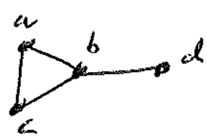
st $x_{i-1} \neq x_{i+1}$ $1 \leq i \leq \ell-1$ "no back tracking allowed"

Define $A \in Mat_X(\mathbb{F})$ with entries

$$A_{xy} = \begin{cases} 1 & \text{if } x, y \text{ adjacent} \\ 0 & \text{if } x, y \text{ not adjacent} \end{cases} \qquad x, y \in X$$

"adjacency matrix"

Ex

Γ : 

A :
$$\begin{array}{c c} & \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{array}{c} a \\ b \\ c \\ d \end{array} & \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

For $x \in X$

$$A\hat{x} = \sum_{y \in \Gamma(x)} \hat{y}$$

For $u, v \in V$ write

$$u = \sum_{x \in X} u_x \hat{x} \qquad v = \sum_{x \in X} v_x \hat{x}$$

$Au = v$ iff

$$\forall x \in X, \qquad v_x = \sum_{y \in \Gamma(x)} u_y$$

Obs $A^t = A$ (symmetric)

$\bar{A} = A$

So A is Hermitean, hence diagonalizable

By the <u>spectrum of Γ</u> we mean $\mathrm{Spec}(A)$

Ex For $n \geq 1$ the <u>complete graph</u> K_n has n vertices with any two adjacent

K_1 K_2 K_3 K_4

$$\mathrm{Spec}(K_n) = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}$$ □

For a graph $\Gamma = (X, R)$

For $x, y \in X$ and $\ell \geq 0$

$(A^\ell)_{xy}$ = the number of walks of length ℓ from x to y.

A walk $x_0, x_1, \ldots, x_\ell$ in Γ is <u>closed</u> whenever $x_0 = x_\ell$

A <u>cycle</u> in Γ is a closed walk $x_0, x_1, \ldots, x_\ell$ with $\ell \geq 3$ and $\{x_i\}_{i=1}^{\ell}$ mutually distinct.

LEM 1 For a graph $\Gamma = (X, R)$ with

Spectrum $\begin{pmatrix} \theta_0 & \theta_1 & \dots & \theta_r \\ m_0 & m_1 & \dots & m_r \end{pmatrix}$

For $\ell \geq 0$

$$\sum_{i=0}^{r} m_i \theta_i^\ell = \text{number of closed walks in } \Gamma \text{ that have length } \ell$$

pf

$$\# \text{ closed walks of length } \ell = \sum_{x \in X} (A^\ell)_{xx}$$

$$= \text{trace}(A^\ell)$$

Recall $A = \sum_{i=0}^{r} \theta_i E_i$

So $A^\ell = \sum_{i=0}^{r} \theta_i^\ell E_i$

$$\text{tr}(A^\ell) = \sum_{i=0}^{r} \theta_i^\ell \underbrace{\text{tr}(E_i)}_{= m_i}$$

$\square$