

Combinatorics 475

Lecture 19

6.6 Mobius inversion

For an integer $n \geq 1$ define

$\phi(n) = \#$ of integers among $1, 2, \dots, n$ that are
relatively prime to n

no prime factor in common with n

"Euler ϕ -function"

ex $n = 6$

1 ~~2~~ ~~3~~ 4 5 ~~6~~

$$\phi(6) = 2$$

For $n \geq 1$ find $\phi(n)$ by inclusion:
Consider prime factorization of n

Write

$$n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$$

$p_1, p_2, \dots, p_k$ distinct
primes

n_i pos integer ($1 \leq i \leq k$)

$$\text{Let } S = \{1, 2, \dots, n\}$$

$$\text{For } 1 \leq i \leq k \quad \text{let}$$

$$A_i = \text{set of integers among } 1, 2, \dots, n \text{ that are divisible by } p_i$$

$$\subseteq S$$

then

$$\phi(n) = \left| \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k} \right|$$

For each subset

$$\Delta \subseteq \{1, 2, \dots, k\}$$

define

$$A_\Delta = \bigcap_{i \in \Delta} A_i$$

and

$$p_\Delta = \prod_{i \in \Delta} p_i$$

$$A_\Delta = \text{set of integers among } 1, 2, \dots, n \text{ that are divisible by } p_\Delta$$

$$= \left\{ p_\Delta, 2p_\Delta, 3p_\Delta, \dots, \frac{n}{p_\Delta} p_\Delta \right\}$$

so

$$|A_\Delta| = \frac{n}{p_\Delta}$$

By inclusion-exclusion

$$\begin{aligned} \phi(n) &= \sum_{\Delta \subseteq \{1, 2, \dots, k\}} |A_\Delta| (-1)^{|\Delta|} \\ &= n \sum_{\Delta \subseteq \{1, 2, \dots, k\}} p_\Delta^{-1} (-1)^{|\Delta|} \\ &= n \prod_{i=1}^k (1 - p_i^{-1}) \end{aligned}$$

This gives

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Thm For $n \geq 1$, the Euler ϕ -function satisfies

$$\phi(n) = n \prod_{i=1}^k (1 - p_i^{-1})$$

where $p_1, p_2, \dots, p_k$ are the distinct primes that divide n

Ex

- $\phi(1) = 1$

- $\phi(p^r) = p^r - p^{r-1} \quad p \text{ prime} \quad r \geq 1$

- For rel prime integers $m, n \geq 1$

$$\phi(mn) = \phi(m) \phi(n)$$

LEM $\forall n \geq 1$

$$n = \sum_{d|n} \phi(d)$$

where the sum is over all pos integers d
that divide n

$$\left[\begin{array}{cccc} \text{ex } 6 = & \phi(1) & + & \phi(2) & + & \phi(3) & + & \phi(6) \\ & \text{"} & & \text{"} & & \text{"} & & \text{"} \\ & 1 & & 1 & & 2 & & 2 \end{array} \right]$$

pf Define

$$R_n = \{ k \mid 1 \leq k \leq n, \quad k, n \text{ rel prime} \}$$

So $\phi(n) = |R_n|$

Obs $\{1, 2, \dots, n\} = \bigcup_{d|n} \frac{n}{d} R_d \quad (\text{disj union})$

So

$$\begin{aligned} n &= |\{1, 2, \dots, n\}| \\ &= \sum_{d|n} |R_d| \\ &\quad \text{" } \phi(d) \end{aligned}$$

□

We now generalize prev lem.

Given any real valued function F defined on pos integers.

Define

$$G(n) = \sum_{d|n} F(d) \quad n \geq 1$$

Find F in terms of G

Case $n = p^r$ p prime $r \geq 0$

divisors d of n are $1, p, p^2, \dots, p^r$

r	$G(p^r)$
0	$F(1)$
1	$F(1) + F(p)$
2	$F(1) + F(p) + F(p^2)$

So

$$F(1) = G(1)$$

$$F(p^r) = G(p^r) - G(p^{r-1}) \quad r \geq 1$$

Case $n = p^r q^s$

p, q dist primes $r, s \geq 1$

$$\begin{aligned} G(n) &= \sum_{d|n} F(d) \\ &= \sum_{i=0}^r \sum_{j=0}^s F(p^i q^j) \end{aligned}$$

One checks

$$F(p^r q^s) = G(p^r q^s) - G(p^{r-1} q^s) - G(p^r q^{s-1}) + G(p^{r-1} q^{s-1})$$

ie

$$F(n) = G(n) - G(n/p) - G(n/q) + G(n/pq)$$

gen case

$$n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

 $p_1, p_2, \dots, p_k$ dist primespos ints $n_1, n_2, \dots, n_k$ $\forall \Delta \subseteq \{1, 2, \dots, k\}$ write

$$p_\Delta = \prod_{i \in \Delta} p_i$$

One checks by incl/excl

$$F(n) = \sum_{\Delta \subseteq \{1, 2, \dots, k\}} G(n/p_\Delta) (-1)^{|\Delta|}$$

*

We now make * more elegant

Define a function μ on set of pos integers:

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n = p_1 p_2 \dots p_k \\ 0 & \text{else} \end{cases}$$

$p_1, p_2, \dots, p_k$
dist primes

then * becomes

$$F(n) = \sum_{d|n} G(n/d) \mu(d)$$

" Mobius inversion "

$$= \sum_{d|n} G(d) \mu(n/d)$$

 μ called Mobius function

ex

Recall $\phi(n)$

$$n = \sum_{d|n} \phi(d)$$

 ϕ = Euler ϕ -function

Def

$$F(n) = \phi(n)$$

$$G(n) = n$$

So

$$G(n) = \sum_{d|n} F(d)$$

By Mobius inversion

$$F(n) = \sum_{d|n} G(d) \mu(n/d)$$

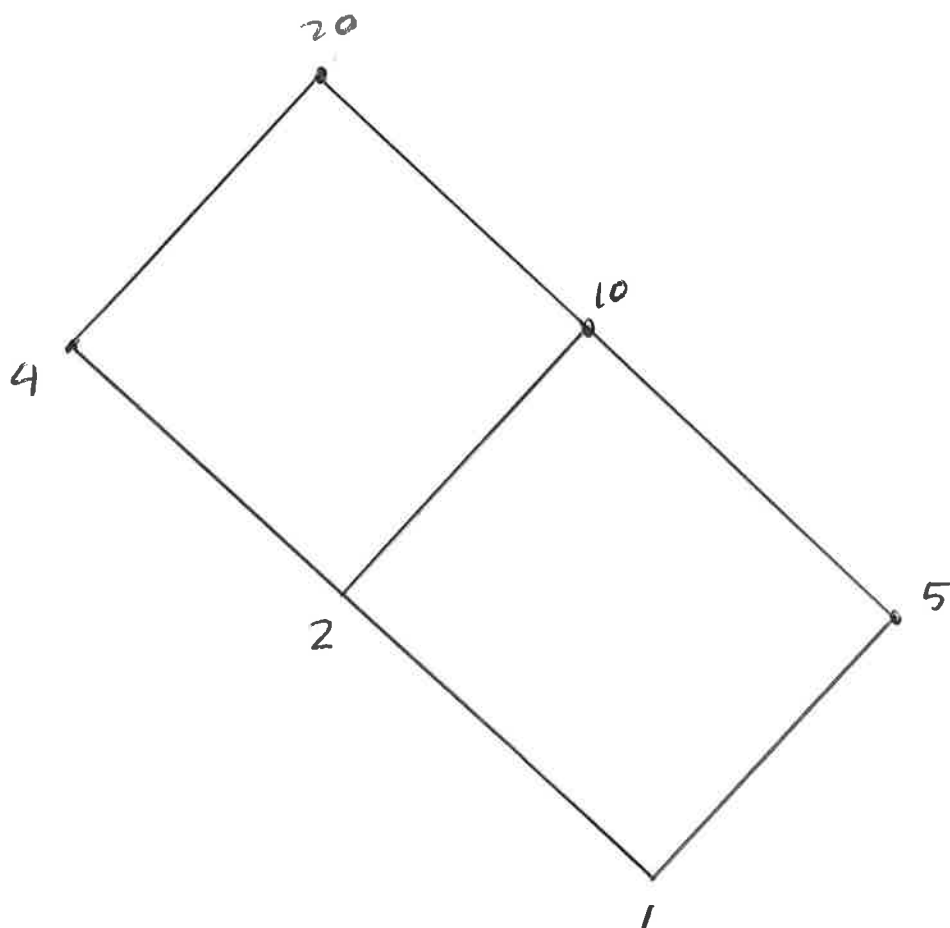
ie

$$\phi(n) = \sum_{d|n} d \mu(n/d)$$

□

Ex Consider $n = 20 = 2 \cdot 2 \cdot 5$

divisors of 20



Find $\mu(d)$ for all divisors d of 20

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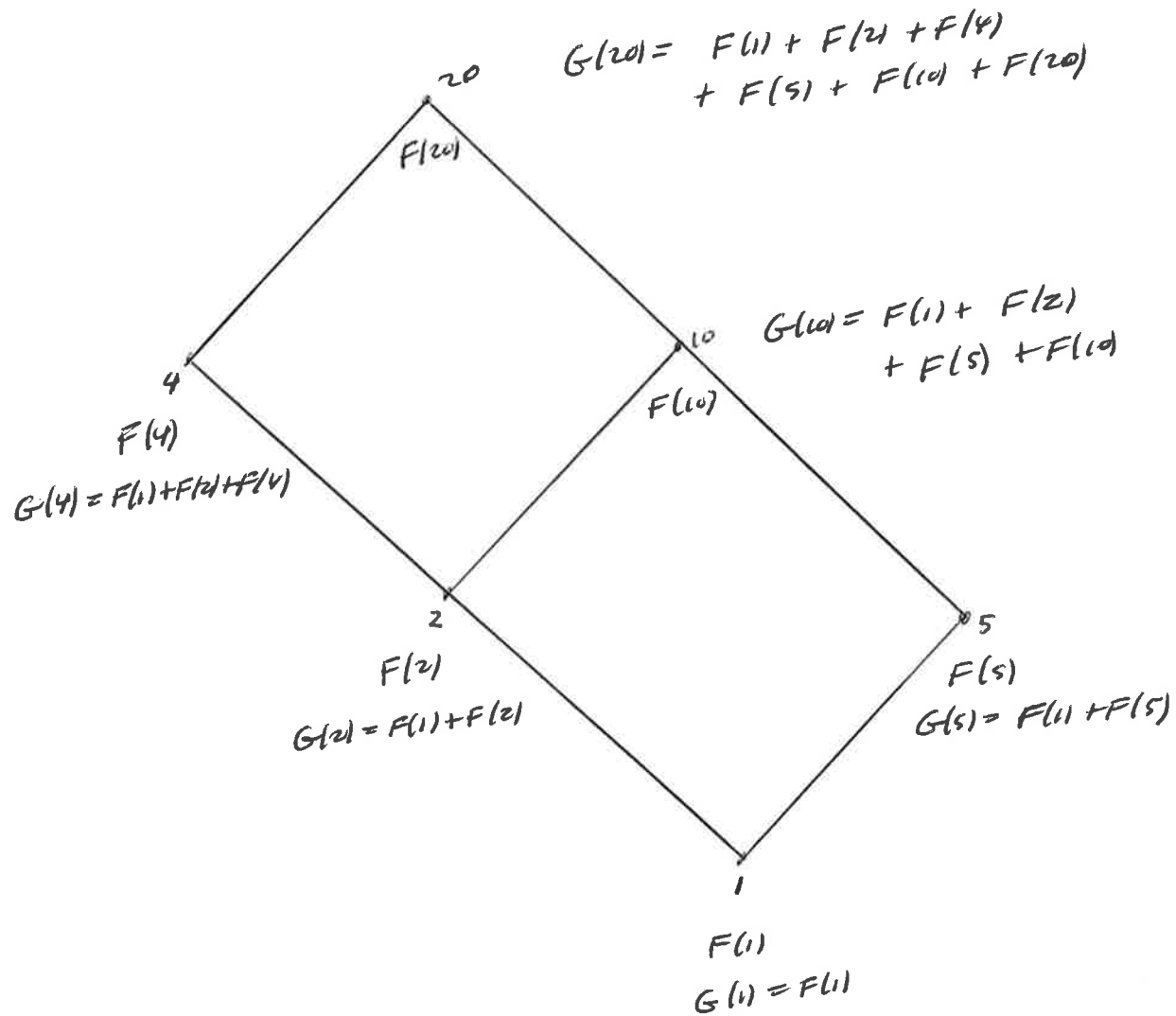
d	factorization	$\mu(d)$
1	1	$(-1)^0 = 1$
2	2	-1
4	$2 \cdot 2$	0
5	5	-1
10	$2 \cdot 5$	$(-1)^2 = 1$
20	$2 \cdot 2 \cdot 5$	0

For any function F ,

Consider $F(d)$,

$G(d)$ for all divisors d of 20

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Expect, by Mobius inversion,

$$F(20) \stackrel{?}{=} \sum_{d|20} G(d) \mu\left(\frac{20}{d}\right)$$

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d	$\frac{20}{d}$	$\mu\left(\frac{20}{d}\right)$	$G(d)$
1	20	0	$F(1)$
2	10	1	$F(1) + F(2)$
4	5	-1	$F(1) + F(2) + F(4)$
5	4	0	$F(1) + F(5)$
10	2	-1	$F(1) + F(2) + F(5) + F(10)$
20	1	1	$F(1) + F(2) + F(4) + F(5) + F(10) + F(20)$

OK

