Math 846

Lecture 19

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter D (not nec Q-polynomial)

Fix $x \in X$ and write $T = T(x)$

LEM 101 For $0 \le i, j \le D$

(i) $$A_i E_j^* V \subseteq \sum_{\substack{0 \le h \le D \\ p_{ij}^h \ne 0}} E_h^* V$$

(ii) $$A_i^* E_j V \subseteq \sum_{\substack{0 \le h \le D \\ q_{ij}^h \ne 0}} E_h V$$

pf (i)
$$
\begin{aligned}
A_i E_j^* V &= I A_i E_j^* V \\
&= \sum_{h=0}^{D} \underbrace{E_h^* A_i E_j^*}_{\substack{\| \text{ if } p_{ij}^h = 0 \\ 0}} V \\
&= \sum_{\substack{0 \le h \le D \\ p_{ij}^h \ne 0}} E_h^* A_i E_j^* V \\
&\subseteq \sum_{\substack{0 \le h \le D \\ p_{ij}^h \ne 0}} E_h^* V
\end{aligned}
$$

(ii) Similar. $\square$

Cor 102 Assume that the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial.

Then

$$A^* E_i V \subseteq E_{i-1}V + E_iV + E_{i+1}V \quad (0 \le i \le D)$$

where $E_{-1} = 0$ and $E_{D+1} = 0$.

pf Set $i=1$ and $A^* = A_1^*$ in Lem 101 (ii). $\square$

LEM 103 Assume the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial. Then T is generated by A and A^*.

pf T is gen by M and M^*

M is gen by A.

M^* is gen by A^*.

$\square$

(Aside) Research problem

Here is another view of the Q-polynomial property.

Let $\Gamma = (X, R)$ denote any connected graph with diameter D. Let $\{E_i\}_{i=0}^{r}$ denote an ordering of the primitive idempotents of Γ.

Fix $x \in X$ and write $M^* = M^*(x)$.

Define a <u>dual adjacency matrix wrt x</u> to be a diagonal matrix $A^* \in \mathrm{Mat}_X(\mathbb{F})$ st

(i) A^* generates M^*

(ii) For $0 \le i \le r$,

$$A^* E_i V \subseteq E_{i-1}V + E_i V + E_{i+1} V$$

where $E_{-1} = 0$ and $E_{r+1} = 0$.

Call Γ <u>Q-polynomial wrt x</u> whenever Γ has at least one dual adjacency matrix wrt x.

Investigate the graphs Γ that are Q-polynomial wrt some vertex. Find examples that are not DRGs.

Chapter 3. The subconstituent algebra of a Q-polynomial distance-regular graph.

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Until further notice, $\Gamma = (X, R)$ is a DRG with diameter $D \geq 1$. Assume Γ is Q-polynomial wrt $\{E_i\}_{i=0}^{D}$. Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Fix $x \in X$ and write $T = T(x)$ etc.

Recall:

- T is generated by A and A^*
- the eigenspaces of A are $\{E_i V\}_{i=0}^{D}$
- For $0 \leq i \leq D$
  $\theta_i =$ eigenvalue of A for $E_i V$
- the eigenspaces of A^* are $\{E_i^* V\}_{i=0}^{D}$
- For $0 \leq i \leq D$
  $\theta_i^* =$ eigenvalue of A^* for $E_i^* V$

- For $0 \leq i \leq D$,

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V$$

where $E_{-1}^* = 0$ and $E_{D+1}^* = 0$

- For $0 \leq i \leq D$

$$A^* E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V$$

where $E_{-1} = 0$ and $E_{D+1} = 0$

Our next goal is to show that A, A^* satisfy a pair of relations called the tridiagonal relations.

We will also obtain a recurrence satisfied by the eigenvalues and dual eigenvalues.

We now state our results.

Thm 1 With above notation, $\exists$ scalars $\beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ such that

$$0 = \left[A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*\right], \qquad \text{TD1}$$

$$0 = \left[A^*,\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^*A\right] \qquad \text{TD2}$$

"Tridiagonal Relations"

Note $[r, s]$ means $rs - sr$.

Thm 2 With above notation, the scalars

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i}, \qquad \frac{\theta^*_{i-2} - \theta^*_{i+1}}{\theta^*_{i-1} - \theta^*_i}$$

are equal and independent of i for $2 \leq i \leq d-1$.

We will prove Thm 1, Thm 2 shortly.

The relations TD1, TD2 can be simplified by applying an affine transformation

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$$A \to rA + sI$$
$$A^* \to r^*A^* + s^*I$$

$r, r^*, s, s^* \in \mathbb{F}$
$r \neq 0, \ r^* \neq 0$

This can be used to normalize TD1, TD2.
We mention three normalizations.

- For $\beta = 2, \ \gamma = 0, \ \gamma^* = 0, \ \delta = 4, \ \delta^* = 4$ the TD1, TD2 become

$$[A, [A, [A, A^*]]] = 4[A, A^*],$$
$$[A^*, [A^*, [A^*, A]]] = 4[A^*, A].$$

"Dolan / Grady relations"

These are the defining relations for the Onsager Lie algebra O. This Lie algebra is used in the statistical mechanics of the Ising model

For the Hamming graph $H(D, N)$ the matrices

$\frac{2A}{N}, \quad \frac{2A^*}{N}$ satisfy the Dolan/Grady rels.

For $q \in \mathbb{F}$ $\quad q \neq 0, 1, -1$

define

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n \in \mathbb{N}$$

"q-integer"

For elements r, s in any algebra define

$$[r, s]_q = q r s - q^{-1} s r$$

Note that

$$[r, [r, [r, s]_q]_{q^{-1}}] =$$

$$r^3 s - [3]_q r^2 s r + [3]_q r s r^2 - s r^3$$

- For $\beta = q^2 + q^{-2}$, $\gamma = 0$, $\gamma^* = 0$, $\delta = 0$, $\delta^* = 0$

the TD1, TD2 become

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = 0,$$

$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = 0$$

"q- Serre relations"

These are the defining relations for the positive part U_q^+ of the quantum group $U_q(\hat{sl}_2)$.

For the bilinear forms graph $H_q(D, N)$ an affine trans of A, A^* satisfy the q-Serre relations.

- For $\beta = q^2 + q^{-2}$, $\gamma = 0$, $\gamma^* = 0$,
$\delta = -(q^2 - q^{-2})^2$, $\delta^* = -(q^2 - q^{-2})^2$

the TD1, TD2 become

$$[A, [A, [A, A^*]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A^*, A]$$

$$[A^*, [A^*, [A^*, A]_q]_{q^{-1}}] = (q^2 - q^{-2})^2 [A, A^*]$$

"q-Dolan/Grady relations"

These are the defining relations for the q-Onsager algebra O_q. This algebra is used in modern statistical mechanics.

Up to affine transformation, the q-Dolan/Grady relations are the "most general" case of the tridiagonal relations.