

Combinatorics 475

Lecture 18

6.4 Perms with forbidden positions

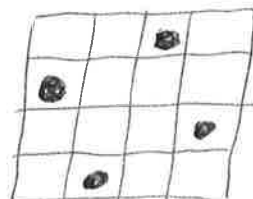
Given perm $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$

View as placing n nonattacking rooks on $n \times n$ chessboard
 i th rook in row i , column a_i ($1 \leq i \leq n$)

ex $n=4$

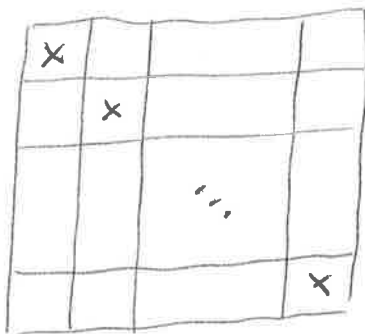
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$\leftrightarrow$



row i	1	2	3	4
col a_i	3	1	4	2

Rook placement is a derangement whenever the
 rooks are not in the forbidden positions X:



We now consider other forbidden positions

To illustrate the technique we consider
small example

Ex Find the number of ways to place 3
nonattacking rooks on the following 3×3 chessboard
with forbidden positions

x	x	
	x	x
		x

Sol We can see ans = 1

But let us solve this using incl/excl

Let S = set of all nonattacking rook placements
ignoring forb positions

$$|S| = 3!$$

Let A_i = rook placements in S where rook i is in a
forb position from i

We seek

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3|$$

By incl/excl

$$|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| = |S| - |A_1| - |A_2| - |A_3|$$

$$+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| - |A_1 \cap A_2 \cap A_3|$$

set X	desc of $ X $	$ X $
S		$3! = 6$
A_1	# x 's in row 1 times $2!$	4
A_2	# x 's in row 2 times $2!$	4
A_3	# x 's in row 3 times $2!$	2
$A_1 \cap A_2$	(# ways to put non attacking rooks in X positions of rows 1, 2) times $1!$	3
$A_1 \cap A_3$	(# ways to put non attacking rooks in X -pos of rows 1, 3) times $1!$	2
$A_2 \cap A_3$	(# ways to put non attacking rooks in X -pos of rows 2, 3) times $1!$	1
$A_1 \cap A_2 \cap A_3$	# ways to put non attacking rooks in X -pos of rows 1, 2, 3	1

$$\begin{aligned}
 |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= 6 - 4 - 4 - 2 + 3 + 2 + 1 - 1 \\
 &= 6 - 10 + 6 - 1 \\
 &= 1
 \end{aligned}$$

Consider partial sums
 $|S|$

$$\begin{aligned} |A_1| + |A_2| + |A_3| &= 3! \\ &= (\text{total \# x's}) 2! \\ &= (\text{\# ways to put 1 rook in x-pos}) 2! \end{aligned}$$

$$|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|$$

$$= (\text{\# ways to put 2 nonattacking rooks in x-pos}) 1!$$

$$|A_1 \cap A_2 \cap A_3| = (\text{\# ways to put 3 nonattacking rooks in x-pos}) 0!$$

For $0 \leq i \leq 3$ define

$r_i = \text{\# ways to put } i \text{ nonattacking rooks in x-pos}$

$$\text{So } r_0 = 1$$

Then

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3| &= r_0 3! - r_1 2! + r_2 1! - r_3 0! \\ &= \sum_{i=0}^3 r_i (3-i)! (-1)^i \end{aligned}$$

General case

Given $n \times n$ chessbd with forbidden positions

Find # ways to put n nonattacking rooks on chessbd, not in forbidden positions.

For $0 \leq i \leq n$ let $r_i = \#$ ways to place i nonattacking rooks in the forbidden positions

then by incl/excl

ways to place n nonattacking rooks on chessbd not in forb positions

$$= \sum_{i=0}^n r_i (n-i)! (-1)^i$$

Ex Find # ways to place 5

nonattacking rooks on chessbd

	1	2	3	4	5
1	X			X	
2			X		
3					
4	X				X
5		X			X

Sol.

For $0 \leq i \leq 5$ find r_i

$$r_0 = 1$$

$$r_1 = \# \text{ X's} = 7$$

$$r_2 = 4 + 5 + 4 + 2 + 1 \\ = 16$$

sim

$$r_3 = 14$$

$$r_4 = 4$$

$$r_5 = 0$$

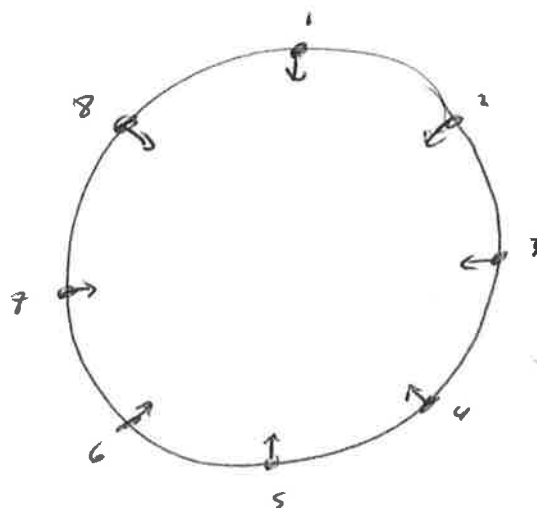
$$\text{Ans} = 5! - 7 \cdot 4! + 16 \cdot 3! - 14 \cdot 2! + 4 \cdot 1!$$

□

Ex A Carousel has 8 seats
each represented by different animal
8 Children are seated on the carousel facing inward
so each child looks at another child's front

In how many ways can children change seats
so each faces a different child?

Sol. Label seats around carousel



Seats labelled $i, i+4$ are opposite $1 \leq i \leq 4$

For $1 \leq i \leq 8$ child i moves to new seat labelled s_i

$s_1, s_2, \dots, s_8$ is perm of $\{1, 2, \dots, 8\}$ s.t. s_i, s_{i+4} are not

opp for $1 \leq i \leq 4$

We comp # such perms

Let $P =$ set of all perms of $\{1, 2, \dots, 8\}$

For $1 \leq i \leq 4$ let $A_i =$ set of perms in P s.t. A_i, A_{i+4} are opp

We seek

$$|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$$

For $S \subseteq \{1, 2, 3, 4\}$ def

$$A_S = \bigcap_{i \in S} A_i$$

We routinely find

$ S $	0	1	2	3	4
$ A_S $	$8!$	$8 \times 6!$	$8 \times 6 \times 4!$	$8 \times 6 \times 4 \times 2!$	$8 \times 6 \times 4 \times 2$

So by incl/excl

$$\text{ans} = |\overline{A_1} \cap \overline{A_2} \cap \overline{A_3} \cap \overline{A_4}|$$

$$= \sum_{S \subseteq \{1, 2, 3, 4\}} |A_S| (-1)^{|S|}$$

$$= 1 \cdot 8! - 4 \cdot 8 \cdot 6! + 6 \cdot 8 \cdot 6 \cdot 4! - 4 \cdot 8 \cdot 6 \cdot 4 \cdot 2! + 1 \cdot 8 \cdot 6 \cdot 4 \cdot 2$$

$$= 23040$$

□

6.5 Another forbidden position problem

Recall Given $n \times n$ chessboard with forbidden positions $\times$

$\forall n, 0 \leq i \leq n$ let $r_i = \#$ ways to place i nonattacking
rooks on $\times$ -positions

Then

$\#$ ways to place n nonattacking rooks on chessboard
not on $\times$ -pos

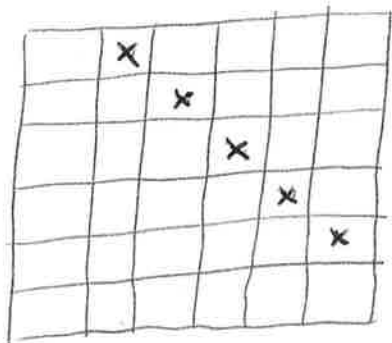
$$= \sum_{i=0}^n r_i (n-i)! (-1)^i$$

Note the r_i are unchanged if we permute the
rows/columns of the chessboard

Ex let $Q_n = \#$ ways to put n nonattacking rooks
on $n \times n$ chessboard with forbidden pos

$$(i, i+1) \quad 1 \leq i \leq n-1$$

$\forall n=6$



A bit easier if we permute rows of chessboard

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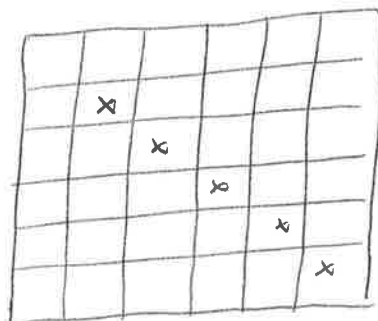
$$1 \rightarrow 2 \rightarrow 3 \rightarrow \dots \rightarrow n \rightarrow 1$$

Forbidden positions become

$$(i, i) \quad 2 \leq i \leq n$$

(X)

$n=6$



For $0 \leq i \leq n$ find r_i :

$r_i =$ # ways to place i nonattacking rooks in X -pos

$$= \begin{cases} \binom{n-1}{i} & \text{if } i \leq n-1 \\ 0 & \text{if } i = n \end{cases}$$

So

$$\Phi_n = \sum_{i=0}^{n-1} \binom{n-1}{i} (n-i)! (-1)^i$$

LEM F_n $n \geq 1$

$$Q_n = P_n + P_{n-1}$$

pf 1: Use above formula for Q_n and earlier formula for P_n

pf 2 (combinatorial)

Consider placement of n rooks using view *

Case rook in row 1 col 1

ways to place remaining $n-1$ rooks is P_{n-1}

Case no rook in row 1 col 1

choices is P_n

□

