Math 846

Lecture 18

Exercise III

Recall the Hamming graph $\Gamma = H(D,N)$

Show that Γ is Q-polynomial with respect to the eigenvalue ordering

$$\theta_0 > \theta_1 > \cdots > \theta_D.$$

For this Q-polynomial structure.

$$q^h_{ij} = p^h_{ij} \qquad 0 \le h,i,j \le D$$

and for $0 \le i \le D$

$$\theta^*_i = \theta_i$$

$$m_i = k_i$$

$$u^*_i = u_i$$

$$v^*_i = v_i$$

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter D. For now, we do not assume Γ is Q-polynomial.

Next goal: A characterization of the Q-polynomial property in terms of the function algebra $V, \circ$.

Fix $x \in X$ and write $T = T(x)$

LEM 96 With above notation, TFAE

(i) the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial

(ii) For $0 \le h, j \le D$

$$q^h_{1j} = \begin{cases} 0 & \text{if } h > j+1 \\ \neq 0 & \text{if } h = j+1 \end{cases}$$

(iii) $\exists$ polynomials $\{v_i^*\}_{i=0}^{D}$ in $\mathbb{F}[\lambda]$ such that

$$\deg v_i^* = i \quad (0 \le i \le D)$$
$$A_i^* = v_i^*(A^*) \quad (0 \le i \le D)$$

pf Use LEM 64. $\square$

Referring to the function algebra $V, \circ$

For a subspace $W \subseteq V$ consider the subalgebra of V generated by W.

This subalgebra contains $\mathbb{1} = \sum_{y \in X} \hat{y}$ by the definition of subalgebra.

To see what else is in the subalgebra, define a binary relation $\sim \; = \; \underset{W}{\sim}$ on X by

$y \sim z$ whenever for all $w \in W$,

y coord of w = z coord of w

Obs $\sim$ is an equiv relation.

For a subset $Y \subseteq X$ define

$$\hat{Y} = \sum_{y \in Y} \hat{y}$$ "characteristic vector" of Y

LEM 97 With above notation, the following are equal:

(i) the subalgebra of V generated by W

(ii) $\mathrm{Span}\{ \hat{Y} \mid Y$ is an equivalence class of $\sim \}$

pf $\subseteq$: $\mathrm{Span}\{ \hat{Y} \mid Y$ is an equiv class of $\sim \}$

is a subalgebra of V that contains W.

$\supseteq$: For an equiv class Y of $\sim$ show

$\hat{Y}$ is in the subalgebra of V generated by W.

List the equivalence classes of $\sim$:

$Y = Y_0, Y_1, Y_2, \ldots, Y_t$

For $w \in W$ write

$$w = \sum_{i=0}^{t} \alpha_i(w) \hat{Y}_i \qquad \alpha_i(w) \in \mathbb{F}$$

For $1 \le i \le t$ $\exists$ $w_i \in W$ such that

$\alpha_0(w_i) \neq \alpha_i(w_i)$

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We have

$$\hat{\varphi} = \prod_{i=1}^{t} \frac{w_i - \alpha_i(w_i)\mathbb{1}}{\alpha_0(w_i) - \alpha_i(w_i)}$$

where the product is with respect to σ.

Result follows. $\square$

Given a primitive idempotent E of Γ.

We now consider the previous discussion for $W = EV$.

Recall

$$EV = \mathrm{Span}(E\hat{y} \mid y \in X)$$

Also for $y, z \in X$

$$
\begin{aligned}
y\text{-coord of } E\hat{z} &= \langle \hat{y}, E\hat{z} \rangle \\
&= \langle E\hat{y}, E\hat{z} \rangle \\
&= \langle E\hat{y}, \hat{z} \rangle \\
&= z\text{-coord of } E\hat{y}.
\end{aligned}
$$

LEM 98 For a primitive idempotent E of Γ, the following are equivalent for all $y, z \in X$:

(i) $y \underset{EV}{\sim} z$

(ii) $E\hat{y} = E\hat{z}$

pf observe

$$y \underset{EV}{\sim} z$$

iff

$$\begin{array}{ccc} y\text{-coord of } E\hat{w} & = & z\text{-coord of } E\hat{w} \\ \| & & \| \\ w\text{-coord of } E\hat{y} & & w\text{-coord of } E\hat{z} \end{array} \quad \forall w \in X$$

iff $E\hat{y} = E\hat{z}$ $\square$

COR 99 For a primitive idempotent E of Γ, the following are equivalent:

(i) EV generates V in the function algebra $V, \circ$

(ii) E is nondegenerate.

pf

EV generates V

$\Longleftrightarrow$ $V = \mathrm{Span}\{ \hat{Y} \mid Y \text{ an equiv class of } \widetilde{EV} \}$

$\Longleftrightarrow$ each equiv class of $\widetilde{EV}$ has single element

$\Longleftrightarrow$ $\{ E\hat{y} \mid y \in X \}$ are mutually distinct

$\Longleftrightarrow$ E is nondegenerate

$\square$

Thm 100 For a DRG $\Gamma = (X, R)$ with diameter $D \geq 1$ and primitive idempotents $\{E_i\}_{i=0}^{D}$.

The following are equivalent:

(i) the ordering $\{E_i\}_{i=0}^{D}$ is Q-polynomial

(ii) E_1 is nondegenerate and

$$E_1 V \circ E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V \qquad (0 \leq i \leq D)$$

where $E_{-1} = 0$ and $E_{D+1} = 0$

(iii) For $0 \leq i \leq D$,

$$E_0 V + E_1 V + \cdots + E_i V = \underbrace{(E_0 V + E_1 V) \circ (E_0 V + E_1 V) \circ \cdots \circ (E_0 V + E_1 V)}_{i \text{ factors}}$$

pf (i) → (ii) Recall the dual eigenvalues

$$\theta_i^* = m_1 u_i(\theta_1) \qquad (0 \le i \le d)$$

$\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct

So $\qquad \theta_i^* \neq \theta_0^* \qquad (1 \le i \le D)$

So $\qquad u_i(\theta_1) \neq 1 \qquad (1 \le i \le D)$

So $\qquad E_1$ is nondegenerate

Also, recall

$$E_1 V \circ E_i V = \sum_{\substack{0 \le h \le D \\ q_{1i}^h \neq 0}} E_h V$$

and

$$q_{1i}^h = 0 \quad \text{if} \quad |h-i| > 1$$

(ii) → (iii) Recall

$$E_0 V = \mathrm{Span}(\mathbb{1}) \qquad \mathbb{1} = \sum_{x \in X} \hat{x}$$

For $0 \le i \le D$ define

$$S_i = E_0V + E_1V + \cdots + E_iV$$

It suffices to show

$$(E_0V + E_1V) \circ S_i = S_{i+1} \qquad (0 \le i \le D-1). \qquad (*)$$

Let i be given $(0 \le i \le D-1)$. Observe

$$(E_0V + E_1V) \circ S_i = S_i + E_1V \circ S_i$$

By construction and (ii),

$$S_i \subseteq S_i + E_1V \circ S_i \subseteq S_{i+1}.$$

We have

$$S_i \neq S_i + E_1V \circ S_i$$

otherwise $E_1V \circ S_i \subseteq S_i$ which can't happen for $i=0$ and for $i \ge 1$ contradicts the fact that E_1V generates the function algebra V.

By these comments and Th 73,

$$S_i + E_1 V \circ S_i = S_{i+1}.$$

We have shown $(*)$, and the result follows.

$(iii) \to (i)$ For $0 \le j \le D-1$,

$$(E_0 V + E_1 V) \circ (E_0 V + \cdots + E_j V) = E_0 V + \cdots + E_{j+1} V$$

So by Thm 73

$$q^h_{1j} = \begin{cases} 0 & \text{if } h > j+1 \\ \neq 0 & \text{if } h = j+1 \end{cases}$$

Now the ordering $\{E_i\}_{i=0}^D$ is Q-polynomial by LEM 96. $\square$

(Aside) Research problem

Given a connected graph $\Gamma = (X, R)$ of diameter $D \geq 1$ (not nec distance-regular)

Assume that Γ is regular with valency k.

Consider the distinct eigenvalues $\{\theta_i\}_{i=0}^{r}$ of Γ $(\theta_0 = k)$. For $0 \leq i \leq r$ let E_i = primitive idempotent of A for θ_i.

Note that $E_0 V = \text{Span}(\mathbb{1})$ $\qquad \mathbb{1} = \sum_{y \in X} \hat{y}$

Call the ordering $\{E_i\}_{i=0}^{r}$ <u>Q-polynomial</u> whenever

(i) $E_1 V$ generates V in the function algebra $V, \circ$.

(ii) for $0 \leq i \leq r$,

$$E_1 V \circ E_i V \subseteq E_{i-1} V + E_i V + E_{i+1} V$$

where $E_{-1} = 0$ and $E_{r+1} = 0$.

Investigate the algebraic/combinatorial properties of Q-polynomial graphs that are not a DRG. Start by finding examples.