

Combinatorics 475

Lecture 17

6.2 Combinations of multisets

Ex Find the number of integer solutions to

$$x + y + z = 20$$

$$0 \leq x \leq 7, \quad 0 \leq y \leq 8, \quad 0 \leq z \leq 9.$$

Sol Let $S =$ set of all nonnegative integer solutions to $x + y + z = 20$.

Define

$A =$ set of elements in S with $x \geq 8$

$B =$ set of elements in S with $y \geq 9$

$C =$ set of elements in S with $z \geq 10$

We seek

$$|\bar{A} \cap \bar{B} \cap \bar{C}|$$

set X	elements in X satisfy			$ X $
S	$0 \leq x$	$0 \leq y$	$0 \leq z$	$\binom{22}{2}$
A	$8 \leq x$	$0 \leq y$	$0 \leq z$	$\binom{14}{2}$
B	$0 \leq x$	$9 \leq y$	$0 \leq z$	$\binom{13}{2}$
C	$0 \leq x$	$0 \leq y$	$10 \leq z$	$\binom{12}{2}$
$A \cap B$	$8 \leq x$	$9 \leq y$	$0 \leq z$	$\binom{5}{2}$
$A \cap C$	$8 \leq x$	$0 \leq y$	$10 \leq z$	$\binom{4}{2}$
$B \cap C$	$0 \leq x$	$9 \leq y$	$10 \leq z$	$\binom{3}{2}$
$A \cap B \cap C$	$8 \leq x$	$9 \leq y$	$10 \leq z$	0

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = \binom{22}{2} - \binom{14}{2} - \binom{13}{2} - \binom{12}{2} \\ + \binom{5}{2} + \binom{4}{2} + \binom{3}{2}$$

Ex Find the number of 20-combinations
for the multiset

$$\{ 7 \cdot a, 8 \cdot b, 9 \cdot c \}$$

Sol Each 20-combination is a multiset

$$\{ x \cdot a, y \cdot b, z \cdot c \}$$

such that

$$x + y + z = 20$$

$$0 \leq x \leq 7$$

$$0 \leq y \leq 8$$

$$0 \leq z \leq 9$$

of 20-combinations

= # of integer solutions to

$$x + y + z = 20$$

$$0 \leq x \leq 7,$$

$$0 \leq y \leq 8$$

$$0 \leq z \leq 9$$

= Same ans as prev problem

□

6.3 Derangements

We introduce derangements with an example

Ex n people put their business cards into a box.
 After shuffling, each person withdraws one card at random.
 Find the probability that no one gets their card back.

Sol Label people $1, 2, \dots, n$

For $1 \leq i \leq n$ let $c_i =$ card drawn by person i

So $c_1, c_2, \dots, c_n$ is perm of $\{1, 2, \dots, n\}$

Define sample space

$P =$ set of all perms of $\{1, 2, \dots, n\}$

Each outcome in P equally likely

$$|P| = n!$$

Event D is the set of perms $c_1, c_2, \dots, c_n$ in P such that
 $c_i \neq i$ for $1 \leq i \leq n$

Find $\text{Prob}(D)$

$$\text{Prob}(D) = \frac{|D|}{|P|}$$

Find $|D|$.

Use incl/excl

For $1 \leq i \leq n$ define

$A_i =$ set of perms $c_1 c_2 \dots c_n$ in P such that
 $c_i = i$

So

$$D = \bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n$$

For $\Delta \subseteq \{1, 2, \dots, n\}$ define

$$A_\Delta = \bigcap_{i \in \Delta} A_i$$

$=$ set of perms $c_1 c_2 \dots c_n$ in P such that
 $c_i = i$ for all $i \in \Delta$

By incl/excl

$$|D| = \sum_{\substack{\Delta \\ \Delta \subseteq \{1, 2, \dots, n\}}} |A_\Delta| (-1)^{|\Delta|}$$

For $\Delta \subseteq \{1, 2, \dots, n\}$ find $|A_\Delta|$ via 2 stage experiment.

To construct a perm $c_1 c_2 \dots c_n \in A_\Delta$ we proceed in stages:

stage	to do	#choices
1	choose c_i for $i \in \Delta$ [$c_i = i$]	1
2	choose c_i for $i \notin \Delta$ [arb perm of $\{1, 2, \dots, n\} \setminus \Delta$]	$(n - \Delta)!$

$$|A_\Delta| = (n - |\Delta|)!$$

Now

$$|D| = \sum_{\substack{\Delta \\ \Delta \subseteq \{1, 2, \dots, n\}}} |A_\Delta| (-1)^{|\Delta|}$$

$$= \sum_{k=0}^n \sum_{\substack{\Delta \subseteq \{1, 2, \dots, n\} \\ |\Delta| = k}} |A_\Delta| (-1)^{|\Delta|} \stackrel{0^k}{=} (n-k)!$$

$$= \sum_{k=0}^n \binom{n}{k} (n-k)! (-1)^k$$

$$= n! \sum_{k=0}^n \frac{(-1)^k}{k!} \stackrel{\frac{n!}{k!(n-k)!}}{=}$$

$$\text{Prob}(0) = \frac{|D|}{|P|}$$

$$= \sum_{k=0}^n \frac{(-1)^k}{k!}$$

□

Note Ref to prev example
what happens to $\text{Prob}(0)$ as $n \rightarrow \infty$?

Recall from calculus

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$= \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

$$\frac{1}{e} = e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$$

$$\lim_{n \rightarrow \infty} \text{Pr}(0) = \frac{1}{e}$$

— 0 —

[e = Euler constant
 $e^{i\pi} = -1$ $i^2 = -1$
 $e \approx 2.71828\dots$
 sum converges for all x]

Def A perm $c_1, c_2, \dots, c_n$ of $\{1, 2, \dots, n\}$ is a derangement

whenever $c_i \neq i$ for $1 \leq i \leq n$.

define

$$D_n = \# \text{ of derangements of } \{1, 2, \dots, n\}$$

We saw

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

n	1	2	3	4	...
D_n	0	1	2	9	...

Ex Relate D_n, D_{n-1}

Sol

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!}$$

$$D_{n-1} = (n-1)! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!}$$

$$\frac{D_n}{n!} = \sum_{k=0}^n \frac{(-1)^k}{k!}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} + \frac{(-1)^n}{n!}$$

$$= \frac{D_{n-1}}{(n-1)!} + \frac{(-1)^n}{n!}$$

So

$$D_n = n D_{n-1} + (-1)^n$$

$n = 2, 3, \dots$

Ex Relate D_n, D_{n-1}, D_{n-2}

Sol

$$D_n = n D_{n-1} + (-1)^n$$

$$\begin{aligned} D_{n-1} &= (n-1) D_{n-2} + (-1)^{n-1} \\ &= (n-1) D_{n-2} - (-1)^n \end{aligned}$$

Add

$$D_n + D_{n-1} = n D_{n-1} + (n-1) D_{n-2}$$

So

$$D_n = (n-1) (D_{n-1} + D_{n-2})$$

$$n = 2, 3, \dots$$

$$D_0 = 1$$

