Math 846

Lecture 17

Next topic: the Q-polynomial property

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

Given DRG $\Gamma = (X, R)$ diameter D

and primitive idempotents $\{E_i\}_{i=0}^{D}$

DEF 78 The ordering $\{E_i\}_{i=0}^{D}$ is called

<u>Q-polynomial</u> whenever the following (i), (ii)

hold for $0 \le h, i, j \le D$

(i) $q^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $q^h_{ij} \neq 0$ if one of h, i, j is equal to the sum of the other two

— o —

Γ might have $0, 1$ or ≥ 2 Q-polynomial orderings of its primitive idempotents.

DEF 79 The DRG Γ is called <u>Q-polynomial</u> whenever there exists at least one Q-polynomial ordering of the primitive idempotents.

— o —

Until further notice, we assume that Γ is Q-polynomial. Fix a Q-polynomial ordering $\{E_i\}_{i=0}^{D}$ of the primitive idempotents of Γ.

Define

$$c_i^* = q^i_{1\,i-1} \qquad 1 \le i \le D$$

$$a_i^* = q^i_{1\,i} \qquad 0 \le i \le D$$

$$b_i^* = q^i_{1\,i+1} \qquad 0 \le i \le D-1$$

and $c_0^* = 0, \quad b_D^* = 0$

Note that

$$a_0^* = 0, \qquad c_1^* = 1,$$

$$c_i^* > 0 \quad (1 \le i \le D), \qquad b_i^* > 0 \quad (0 \le i \le D-1),$$

$$c_i^* + a_i^* + b_i^* = m_1 \qquad (0 \le i \le D)$$

LEM 80 With above notation,

(i) $m_i c_i^* = m_{i-1}^* b_{i-1}^*$ $(1 \le i \le \rho)$

(ii) $m_i = \dfrac{b_0^* b_1^* \cdots b_{i-1}^*}{c_1^* c_2^* \cdots c_i^*}$ $(0 \le i \le \rho)$

pf (i) this is

$$m_h q_{i\ell}^h = m_i q_{h\ell}^i$$

with $\ell = 1$ and $h = i-1$

(ii) Use (i) above and $m_0 = 1$. $\square$

Fix $x \in X$ and recall

$$A_i^* = A_i^*(x) \qquad E_i^* = E_i^*(x) \qquad (0 \le i \le D)$$

We abbreviate $A^* = A_1^*$ and call this the <u>dual adjacency matrix</u> of Γ with respect to x (and the given Q-poly structure)

LEM 81 With above notation,

$$A^* A_i^* = b_{i-1}^* A_{i-1}^* + a_i^* A_i^* + c_{i+1}^* A_{i+1}^* \qquad (0 \le i \le D)$$

where $A_{-1}^* = 0$, $A_{D+1}^* = 0$, $b_{-1}^* = 1$, $c_{D+1}^* = 1$

pf This is just

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$$

with $j = 1$ $\square$

In Lem 61 we saw

$$A^* = m_1 \sum_{i=0}^{D} u_i(\theta_1) E_i^*$$

Abbrev

$$\theta_i^* = m_1 u_i(\theta_1) \qquad (0 \le i \le D)$$

so that

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

Shortly we will see that $\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct.

DEF 82 With above notation, define polynomials

$\{v_i^*\}_{i=0}^{D+1}$ in $\mathbb{F}[\lambda]$ by

$$v_0^* = 1, \qquad v_1^* = \lambda,$$

$$\lambda v_i^* = c_{i+1}^* v_{i+1}^* + a_i^* v_i^* + b_{i-1}^* v_{i-1}^* \qquad (1 \le i \le D)$$

where $c_{D+1}^* = 1$

LEM 83 We have

(i) $\deg v_i^* = i$ $\quad (0 \le i \le D+1)$

(ii) the coef of λ^i in v_i^* is $(c_1^* c_2^* \cdots c_i^*)^{-1}$ $\quad (0 \le i \le D+1)$

(iii) $v_i^*(A^*) = A_i^*$ $\quad (0 \le i \le D)$

(iv) $v_{D+1}^*(A^*) = 0$

pf (i), (ii) By Def 82

(iii), (iv) Compare Lem 81, Def 82. □

Lec 17
7

LEM 84 With above notation,

(i) A^* generates M^*

(ii) $\{\theta_i^*\}_{i=0}^D$ are mutually distinct

(iii) $\{\theta_i^*\}_{i=0}^D$ are the zeros of v_{D+1}^*

pf (i) By LEM 83 (iii) and since $\{A_i^*\}_{i=0}^D$ is a basis for M^*

(ii) By (i)

(iii) By LEM 83 (iv) v_{D+1}^* is a scalar mult of the minimal poly of A^*

□

DEF 85 With above notation, define

$$B^* = \begin{pmatrix} a_0^* & b_0^* & & & \mathbf{0} \\ c_1^* & a_1^* & b_1^* & & \\ & c_2^* & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1}^* \\ \mathbf{0} & & & c_d^* & a_d^* \end{pmatrix} \qquad D+1 \times D+1$$

Call B^* the <u>dual intersection matrix</u> of Γ (with respect to the given Q-polynomial structure).

LEM 86 The following coincide:

(i) the min polynomial of A^*

(ii) the min polynomial of B^*

(iii) the char polynomial of B^*

pf Similar to LEM 14. $\square$

COR 87 The eigenvalues of B^* are

$$\{\theta_i^*\}_{i=0}^{D}$$

pf By LEM 86 and the construction $\square$

LEM 88 For any eigenvalue $\theta^* = \theta_j^*$ for B^* define a row vector

$$v^* = \left(v_0^*(\theta^*), \; v_1^*(\theta^*), \ldots, v_D^*(\theta^*) \right)$$

where the polynomials v_i^* are from DEF 82. Then

$$v^* B^* = \theta^* v^*$$

pf Similar to LEM 16. $\square$

DEF 89 Define polynomials $\{u_i^*\}_{i=0}^D$ in $\mathbb{F}[\lambda]$ by

$$u_0^* = 1, \qquad u_1^* = \frac{\lambda}{m_1}$$

$$\lambda u_i^* = c_i^* u_{i-1}^* + a_i^* u_i^* + b_i^* u_{i+1}^* \qquad (1 \le i \le D-1)$$

LEM 90 We have

$$u_i^* = \frac{v_i^*}{m_i} \qquad 0 \le i \le D.$$

pf Similar to LEM 18. $\square$

LEM 91 For an eigenvalue $\theta^* = \theta_j^*$ of B^* define a column vector

$$u^* = \begin{pmatrix} u_0^*(\theta^*) \\ u_1^*(\theta^*) \\ \vdots \\ u_D^*(\theta^*) \end{pmatrix}$$

then

$$B^* u^* = \theta^* u^*.$$

pf Similar to LEM 19. $\square$

Thm 92 (Askey–Wilson duality)

With the above notation,

$$u_i(\theta_j) = u_j^*(\theta_i^*) \qquad (0 \le i,j \le D)$$

pf Obs

$$
\begin{aligned}
A_j^* &= A_j^* I \\
&= A_j^* \sum_{i=0}^{D} E_i^* \\
&= v_j^*(A^*) \sum_{i=0}^{D} E_i^* \\
&= \sum_{i=0}^{D} v_j^*(\theta_i^*) E_i^* \\
&= m_j \sum_{i=0}^{D} u_j^*(\theta_i^*) E_i^*
\end{aligned}
$$

Also by LEM 61,

$$A_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j) E_i^*$$

Result follows. $\square$

LEM 93 With above notation

(i) $u_i^*(\theta_0^*) = 1$ $(0 \le i \le D)$

(ii) $v_i^*(\theta_0^*) = m_i$ $(0 \le i \le D)$

pf (i) $u_i^*(\theta_0^*) = u_0(\theta_i)$
$= 1$

(ii) By (i) and since $v_i^* = m_i u_i^*$ $\square$

Earlier we obtained some "row" and "column" orthogonality relations for the polynomials u_i and v_i. Using the same methods we could get similar relations for the polynomials u_i^* and v_i^*. We could also use Askey-Wilson duality.

Thm 94. With above notation

(i) For $0 \le i, j \le D$

$$\sum_{r=0}^{D} u_i^*(\theta_r^*)\, u_j^*(\theta_r^*)\, k_r = \delta_{ij}\, m_i^{-1} |X|$$

(ii) For $0 \le r, s \le D$

$$\sum_{i=0}^{D} u_i^*(\theta_r^*)\, u_i^*(\theta_s^*)\, m_i = \delta_{rs}\, k_r^{-1} |X|$$

pf Combine thm 36 and thm 92. $\square$

Thm 95 With above notation,

(i) For $0 \le i, j \le D$

$$\sum_{r=0}^{D} v_i^*(\theta_r^*)\, v_j^*(\theta_r^*)\, k_r = \delta_{ij}\, m_i\, |X|$$

(ii) For $0 \le r, s \le D$

$$\sum_{i=0}^{D} v_i^*(\theta_r^*)\, v_i^*(\theta_s^*)\, m_i^{-1} = \delta_{rs}\, k_r^{-1}\, |X|$$

pf Evaluate thm 94 using $v_\ell^* = u_\ell^* m_\ell$ for $0 \le \ell \le D$ $\square$