

Combinatorics 475

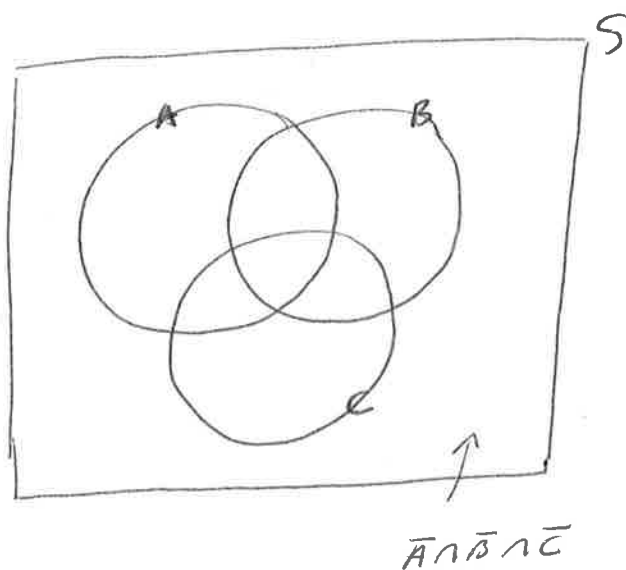
Lecture 16

Ch 6 Inclusion/Exclusion + Applications

6.1 Incl/Excl

Ex Given a finite set S

Given subsets $A, B, C \subseteq S$



Find $|\bar{A} \cap \bar{B} \cap \bar{C}|$

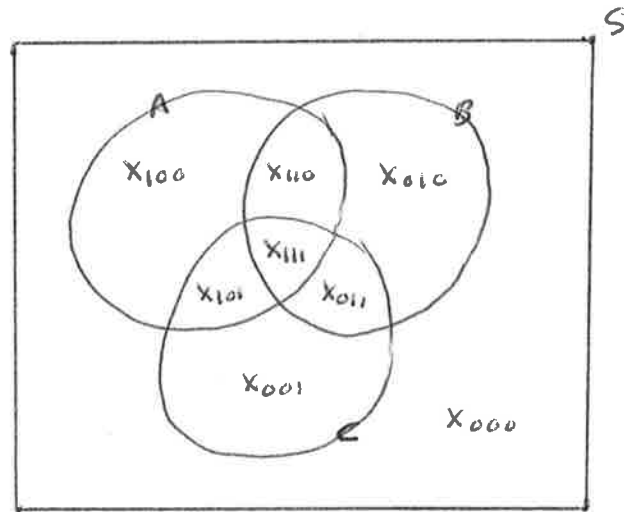
in terms of

$$\begin{aligned} &|S| \\ &|A| \quad |B| \quad |C| \\ &|A \cap B| \quad |A \cap C| \quad |B \cap C| \\ &|A \cap B \cap C| \end{aligned}$$

*

Sol

Define variables x_{ijk} as shown



So $x_{100} = |A \cap \bar{B} \cap \bar{C}|$ etc.

We desire x_{000}

Write $*$ in terms of x_{ijk}

	x_{000}	x_{100}	x_{010}	x_{001}	x_{110}	x_{101}	x_{011}	x_{111}
S	1	1	1	1	1	1	1	1
A	0	1	0	0	1	1	0	1
B	0	0	1	0	1	0	1	1
C	0	0	0	1	0	1	1	1
AND	0	0	0	0	1	0	0	1
ANC	0	0	0	0	0	1	0	1
BNC	0	0	0	0	0	0	1	1
ANBNC	0	0	0	0	0	0	0	1

Each row above gives equation. For instance

$$|C| = x_{001} + x_{101} + x_{011} + x_{111}$$

Now "backsolve" to get the x 's in terms of *

$$X_{111} = |A \cap B \cap C|$$

$$X_{011} = |B \cap C| - |A \cap B \cap C|$$

$$X_{101} = |A \cap C| - |A \cap B \cap C|$$

$$X_{110} = |A \cap B| - |A \cap B \cap C|$$

$$X_{001} = |C| - X_{101} - X_{011} - X_{111}$$

$$= |C| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$X_{010} = |B| - |A \cap B| - |B \cap C| + |A \cap B \cap C|$$

$$X_{100} = |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$

$$X_{000} = |S| - (\text{all the above } X_{ijk})$$

=

term	S	A	B	C	A ∩ B	A ∩ C	B ∩ C	A ∩ B ∩ C
coef	1							-1
							-1	1
						-1		1
					-1			1
			-1		1		1	-1
		1			1	1		-1
coef	1	-1	-1	-1	1	1	1	-1

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = X_{000}$$

$$= |S| - |A| - |B| - |C|$$

$$+ |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$$

□

the above example is "n=3" case of Inclusion/Exclusion

General Case

Given finite set S

Given subsets $A_1, A_2, \dots, A_n$ of S

For each subset $\Delta \subseteq \{1, 2, \dots, n\}$ define

$$A_\Delta = \bigcap_{i \in \Delta} A_i$$

For example $n=6$

Δ	A_Δ
$\{2, 4\}$	$A_2 \cap A_4$
$\{2, 3, 6\}$	$A_2 \cap A_3 \cap A_6$
$\emptyset$	S

Theorem (Inclusion/Exclusion)

With the above notation

$$|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n| = \sum_{\Delta} |A_{\Delta}| (-1)^{|\Delta|} \quad *$$

where the sum is over all the subsets Δ of $\{1, 2, \dots, n\}$

pf Fix $\theta \in S$

Show that the contribution of θ to each side of $*$ is the same.

Define a subset $\Delta \subseteq \{1, 2, \dots, n\}$ by

$$\Delta = \{i / \theta \in A_i\}$$

So $\Delta = \emptyset \iff \theta \in \bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n$

For LHS of $*$:

$$\theta \text{ contributes } \begin{cases} 1 & \text{if } \Delta = \emptyset \\ 0 & \text{if } \Delta \neq \emptyset \end{cases}$$

Consider RHS of $*$:

For $1 \leq i \leq n$

$$\theta \in A_i \quad \text{if } i \in \Delta$$

$$\theta \notin A_i \quad \text{if } i \notin \Delta$$

So for $\Delta \subseteq \{1, 2, \dots, n\}$

$\theta \in A_\Delta$ if $\Delta \subseteq \Delta$ (Here θ contributes 1 to $|A_\Delta|$)

$\theta \notin A_\Delta$ if $\Delta \not\subseteq \Delta$ (... 0 ...)

Contrib of θ to RHS of *

$$= \sum_{\Delta \subseteq \Delta} (-1)^{|\Delta|}$$

$$= \sum_{i=0}^{|\Delta|} \sum_{\substack{\Delta \subseteq \Delta \\ |\Delta|=i}} (-1)^{|\Delta|}$$

$$= \sum_{i=0}^{|\Delta|} \binom{|\Delta|}{i} (-1)^i$$

$$= \begin{cases} (-1)^{|\Delta|} = 0 & \text{if } \Delta \neq \emptyset \\ 1 & \text{if } \Delta = \emptyset \end{cases}$$

So contribution of θ to each side of * is same ✓

□

Ex Find the number of integers among
 $1, 2, 3, \dots, 100$

that are not divisible by 4 or 5 or 6

Sol Define a set

$$S = \{1, 2, \dots, 100\}$$

Define

$$A = \{i \in S \mid 4 \text{ divides } i\}$$

$$B = \{i \in S \mid 5 \text{ divides } i\}$$

$$C = \{i \in S \mid 6 \text{ divides } i\}$$

We seek $|\bar{A} \cap \bar{B} \cap \bar{C}|$

Prime factorization

$$4 = 2 \times 2$$

$$5 = 5$$

$$6 = 2 \times 3$$

set X	elements in X are divisible by
A	4
B	5
C	6
$A \cap B$	$4 \times 5 = 20$
$A \cap C$	$2 \times 2 \times 3 = 12$
$B \cap C$	$5 \times 6 = 30$
$A \cap B \cap C$	$2 \times 2 \times 3 \times 5 = 60$

Set	size	reason
S	100	
A	25	$100 = 4 \times 25$
B	20	$100 = 5 \times 20$
C	16	$100 = 6 \times 16 + 4$
$A \cap B$	5	$100 = 20 \times 5$
$A \cap C$	8	$100 = 12 \times 8 + 4$
$B \cap C$	3	$100 = 30 \times 3 + 10$
$A \cap B \cap C$	1	$100 = 60 \times 1 + 40$

$$6 \times 16 = 96$$

B₄ incl/excl

$$|\bar{A} \cap \bar{B} \cap \bar{C}| = 100 - 25 - 20 - 16 \\ + 5 + 8 + 3 \\ - 1$$

$$= 54$$

□

