Math 846

Lecture 16

We continue to discuss a DRG $\Gamma = (X, R)$ with diam D.

Next goal: a geometric interpretation of the Krein parameters

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Consider standard module $V = \mathbb{F}^X$

Pick $v \in V$

write $$v = \sum_{y \in X} v_y \hat{y} \qquad v_y \in \mathbb{F}$$

View v as a function

$$v: \begin{array}{ccc} X & \longrightarrow & \mathbb{F} \\ y & \longrightarrow & v_y \end{array}$$

For the moment, view V as the set of all functions $X \to \mathbb{F}$

The vector space V, together with product of functions is an $\mathbb{F}$-algebra.

For two vectors in V:

$$v = \sum_{y \in X} v_y \hat{y} \qquad w = \sum_{y \in X} w_y \hat{y}$$

Write

$$v \circ w = \sum_{y \in X} v_y w_y \hat{y}$$

To represent the product of v, w viewed as functions. Fix $x \in X$ and write $T = T(x)$

LEM 72 With above notation, for $0 \le i \le D$ and $v \in V$

$$A_i^* v = |X| \; E_i \hat{x} \circ v.$$

pf For $y \in X$ coord y of $A_i^* v$ is

$$(A_i^* v)_y = (A_i^*)_{yy} v_y$$

$$= |X| \, (E_i)_{xy} \, v_y.$$

Also

$$(E_i \hat{x} \circ v)_y = (E_i \hat{x})_y \, v_y$$

$$= (E_i)_{yx} \, v_y$$

$$= (E_i)_{xy} \, v_y.$$

Result follows. $\square$

<u>Notation</u> For subspaces Y, Z of V define

$$Y \circ Z = \mathrm{Span}\{ y \circ z \mid y \in Y, z \in Z \}$$

Thm 73 With above notation and for $0 \le i, j \le D$

$$E_i V \circ E_j V = \sum_{\substack{0 \le h \le D \\ q_{ij}^h \neq 0}} E_h V.$$

pf $\subseteq$: Given h $(0 \le h \le D)$ st $q_{ij}^h = 0$

show $$E_h \left(E_i V \circ E_j V \right) = 0$$

Suffices to show: $\forall\, y, z \in X$
$$E_h \left(E_i \hat{y} \circ E_j \hat{z} \right) = 0$$

Since our base vertex x is arbitrary, WLOG $x = y$

show $$E_h \left(E_i \hat{x} \circ E_j \hat{z} \right) = 0 \qquad (*)$$

We have
$$E_h \left(E_i \hat{x} \circ E_j \hat{z} \right) = E_h \left(|X|^{-1} A_i^* E_j \hat{z} \right)$$
$$= |X|^{-1} \underbrace{\left(E_h A_i^* E_j \right)}_{\overset{\parallel}{0} \text{ since } q_{ij}^h = 0} \hat{z}$$
$$= 0$$

$\supseteq$: For $0 \le h \le D$ st $q_{ij}^h \neq 0$ show

$$E_i V \circ E_j V \supseteq E_h V.$$

Obs

$$E_i V \circ E_j V = \text{Span}\{ E_i \hat{y} \circ E_j \hat{z} \mid y, z \in X \}$$

$$\supseteq \text{Span}\{ \underbrace{E_i \hat{y} \circ E_j \hat{y}}_{= (E_i \circ E_j)\hat{y}} \mid y \in X \}$$

$$= (E_i \circ E_j) \text{Span}\{ \hat{y} \mid y \in X \}$$

$$= (E_i \circ E_j) V$$

$$\supseteq (E_i \circ E_j) E_h V$$

$$= \left(|X|^{-1} \sum_{\ell=0}^{D} q_{ij}^{\ell} E_\ell \right) E_h V$$

$$= \underset{\neq 0}{|X|^{-1}} \underset{\neq 0}{q_{ij}^h} E_h V$$

$$= E_h V$$

$\square$

Cor 74 For $0 \le i, j \le D$

$$m_i m_j \ge \sum_{\substack{0 \le h \le D \\ q^h_{ij} \ne 0}} m_h$$

pf Recall

$$m_r = \dim E_r V \qquad (0 \le r \le D)$$

Obs

$$\dim\left(E_i V \circ E_j V\right) \le \left(\dim E_i V\right)\left(\dim E_j V\right)$$

Result follows in view of Th 73. $\square$

Cor 74 gives another "feasibility condition" on the intersection numbers

LEM 75 With above notation and for $0 \le j \le D$

define a binary operation

$$* : \quad E_j V \times E_j V \to E_j V$$
$$u \quad v \quad \to u * v$$

where $u * v = E_j(u \circ v)$.

then

(i) $u * v = v * u$ $\quad u, v \in E_j V$

(ii) $u * (v + v') = u * v + u * v'$ $\quad u, v, v' \in E_j V$

(iii) $(\alpha u) * v = \alpha(u * v)$ $\quad \alpha \in \mathbb{F} \quad u, v \in E_j V$

(iv) $u * v = 0 \quad \forall u, v \in E_j V$ iff $q^j_{jj} = 0$

pf (i) – (iii) clear

(iv) By Thm 73

$\square$

Note 76 With reference to Lem 75

we call the vector space $E_j V$ together with $*$

the Norton algebra on $E_j V$.

We denote this algebra by N_j.

The algebra N_j is

commutative, non associative, no mult identity.

By an automorphism of N_j we mean an iso

of vector spaces

$$\sigma : E_j V \to E_j V$$

st

$$\sigma(u * v) = \sigma(u) * \sigma(v) \qquad \forall u, v \in E_j V$$

The set of automorphisms of N_j forms a group

under composition, denoted $\mathrm{Aut}(N_j)$

Next we consider how is $\mathrm{Aut}(N_j)$ related to $\mathrm{Aut}(\Gamma)$.

LEM 77 Given a DRG $\Gamma = (X, R)$ with diameter D. Then for $0 \le j \le D$,

(i) $\sigma|_{E_j V} \in Aut(N_j) \quad \forall \sigma \in Aut(\Gamma)$

(ii) $$\begin{aligned} Aut(\Gamma) &\longrightarrow Aut(N_j) \\ \sigma &\longrightarrow \sigma|_{E_j V} \end{aligned}$$ is hom of groups

(iii) Assume $u_i(\theta_j) \neq 1$ for $1 \le i \le D$ (ie E_j is non degenerate)
then the hom (ii) is injective.

pf (i) σ is invertible on V and

$\sigma E_j V \subseteq E_j V$ so

$\sigma|_{E_j V} : E_j V \to E_j V$ is iso of vector spaces.

Also $\forall\, u, v \in E_j V$

$$\begin{array}{ccc} \sigma(u * v) & \stackrel{?}{=} & \sigma(u) * \sigma(v) \\ \| & & \| \\ \sigma(E_j(u \circ v)) & & E_j(\sigma(u) \circ \sigma(v)) \\ \| & & \\ E_j \sigma(u \circ v) & & \end{array}$$

$\sigma(u \circ v) = \sigma(u) \circ \sigma(v)$ since σ permutes the coords in V

(ii) By construction.

(iii) Each $\sigma \in \mathrm{Aut}(\Gamma)$ permutes the vectors

$$\{ E_y \hat{y} \mid y \in X \},$$

and these vectors are mutually distinct since E_y is nondegenerate.

$\square$

Aside

- The monster finite simple group (Mon) was first constructed around 1980 by Robert Greiss.
- At that time the character table of Mon was known, but it was not known if that table corresponded with an actual group.
- Here is a summary of the construction, due to Greiss.
- There is a mild generalization of a DRG called a commutative association scheme.
- Any finite group G gives a commutative association scheme, called the group scheme Γ_G
- There is a natural injection of groups $G \to Aut(\Gamma_G)$
- Using the character table of Mon, compute the intersection numbers p^h_{ij} and Krein parameters q^h_{ij} of Γ_{Mon}

Aside, cont.

- Find J where m_J is small and $q^J_{JJ} \neq 0$

- Guess abstract structure of Norton algebra N_J
  using p^h_{iJ}, q^h_{iJ}

- Compute $\mathrm{Aut}(N_J)$

- Using the injection

$$M\,m \to \mathrm{Aut}(\Gamma_{Mm}) \to \mathrm{Aut}(N_J)$$

find $M\,m$ as a subgroup of $\mathrm{Aut}(N_J)$.