

Combinatorics 475

Lecture 15

§.5 Newton's binomial theorem

Recall binomial theorem

$$(1+z)^n = \sum_{k=0}^n \binom{n}{k} z^k \quad n = 0, 1, 2, \dots$$

So

$$(1-z)^n = \sum_{k=0}^n \binom{n}{k} (-1)^k z^k$$

Now consider

$$(1-z)^{-n} \quad n = 0, 1, 2, \dots$$

View I

First note

$$\frac{1}{1-z} = 1 + z + z^2 + \dots$$

where z is any real number such that $|z| < 1$

check: let

$$S = 1 + z + z^2 + \dots$$

$$zS = z(1 + z + z^2 + \dots)$$

$$= z + z^2 + \dots$$

$$= S - 1$$

so

$$1 = S(1-z)$$

so

$$S = \frac{1}{1-z}$$

Now

$$\begin{aligned}
 (1-z)^{-n} &= \frac{1}{(1-z)^n} \\
 &= \frac{1}{1-z} \frac{1}{1-z} \cdots \frac{1}{1-z} \\
 &= \left(1+z+z^2+\cdots\right) \left(1+z+z^2+\cdots\right) \cdots \\
 &\quad (n \text{ factors})
 \end{aligned}$$

$$= \sum a_1 a_2 \cdots a_n$$

sum over all products $a_1 a_2 \cdots a_n$ such that $a_i \in \{1, z, z^2, \dots\}$ for $1 \leq i \leq n$ Each summand $a_1 a_2 \cdots a_n$ contributes

$$z^k$$

where

$$k = k_1 + k_2 + \cdots + k_n$$

$$a_i = z^{k_i}$$

$$1 \leq i \leq n$$

For an integer $k \geq 0$ the number of summands $a_1 a_2 \cdots a_n$ that contribute z^k

$$= \# \text{ non neg integral solns } k_1, k_2, \dots, k_n \text{ to}$$

$$k = k_1 + k_2 + \cdots + k_n$$

$$= \binom{k+n-1}{n-1} = \binom{k+n-1}{k}$$

So

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^k$$

$$|z| < 1$$

View II

If we retain the form of binom theorem but allow neg exponents, answer "should be"

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k z^k$$

Check:

$$\binom{-n}{k} (-1)^k \stackrel{?}{=} \binom{n+k-1}{k}$$

$$k = 0, 1, 2, \dots$$

$$\text{LHS} = \frac{(-n)(-n-1) \dots (-n-k+1)}{k!} (-1)^k$$

$$= \frac{n(n+1) \dots (n+k-1)}{k!}$$

$$\text{RHS} = \frac{(n+k-1)(n+k-2) \dots (n+1)n}{k!}$$

$$= \text{LHS} \quad \checkmark$$

So

$$(1-z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} (-1)^k z^k$$

$$n = 0, 1, 2, \dots$$

$$|z| < 1$$

or

$$(1+z)^{-n} = \sum_{k=0}^{\infty} \binom{-n}{k} z^k$$

We have shown:

For an integer n (pos or neg)

$$(1+z)^n = \sum_{k=0}^{\infty} \binom{n}{k} z^k$$

$$|z| < 1$$

In fact much more is true:

thm (Newton) For any real number α

$$(1+z)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} z^k$$

$$|z| < 1$$

[pt requires advanced calculus]

Ex Find $\sqrt{3}$ (decimal expansion)

Sol

$$\sqrt{3} = 3^{1/2}$$

$$= (4-1)^{1/2}$$

$$= \left(4 \left(1 - \frac{1}{4} \right) \right)^{1/2}$$

$$= 2 \left(1 - \frac{1}{4} \right)^{1/2}$$

$$= 2 (1-z)^{1/2}$$

$$z = \frac{1}{4}$$

$$= 2 \sum_{k=0}^{\infty} \binom{1/2}{k} (-1)^k z^k$$

Fn $n=0,1,2,\dots$ def

$$a_n = 2 \sum_{k=0}^n \binom{1/2}{k} (-1)^k z^k$$

$$\lim_{n \rightarrow \infty} a_n = \sqrt{3}$$

For large n

a_n is good approx to $\sqrt{3}$

— o —

5.6 More on posets

Given a nonempty finite set X with partial order $\leq$

Recall

For $\emptyset \neq C \subseteq X$,

C is chain whenever any 2 elements of C are comparable

For $\emptyset \neq A \subseteq X$

A is antichain whenever any 2 distinct elements of C are
incomparable

For a chain C and antichain A

$$|C \cap A| \leq 1$$

Since for distinct $x, y \in C \cap A$

x, y both comparable and incomparable cont

Suppose we partition X into antichains $\{A_i\}_{i=1}^{\alpha}$

So

$$X = \bigcup_{i=1}^{\alpha} A_i \quad (\text{disj union})$$

For all chains C ,

$$\begin{aligned} C &= C \cap X \\ &= C \cap \left(\bigcup_{i=1}^{\alpha} A_i \right) \\ &= \bigcup_{i=1}^{\alpha} (C \cap A_i) \end{aligned}$$

So

$$\begin{aligned} |C| &= \sum_{i=1}^{\alpha} |C \cap A_i| \\ &\leq \alpha \end{aligned}$$

Therefore

$$\max\{|C| \mid C \text{ chain in } X\} \leq \min\{\alpha \mid \{A_i\}_{i=1}^{\alpha} \text{ is antichain partition of } X\}$$

(*)

Thm: equality holds in (*)

pf Suffices to display a chain C of X and

an antichain partition $\{A_i\}_{i=1}^{\alpha}$ of X such that

$$|C| = \alpha.$$

Recall an element $x \in X$ is minimal whenever there does not exist $y \in X$ such that $y < x$.

define

$A_1 =$ set of all minimal elements of X

$A_2 =$ set of all minimal elements of $X \setminus A_1$

$A_3 =$ set of all minimal elements of $X \setminus \{A_1 \cup A_2\}$

...

$A_\alpha \neq \emptyset$ and $A_{\alpha+1} = \emptyset$

for some α

$\{A_i\}_{i=1}^\alpha$ is antichain partition of X

By construction, for $2 \leq i \leq \alpha$ and $x \in A_i$

$\exists y \in A_{i-1}$ such that $y < x$

We now construct a chain C with $|C| = \alpha$

Pick $x_\alpha \in A_\alpha$

$\exists x_{\alpha-1} \in A_{\alpha-1}$ such that

$$x_{\alpha-1} < x_\alpha$$

$\exists x_{\alpha-2} \in A_{\alpha-2}$ such that

$$x_{\alpha-2} < x_{\alpha-1}$$

$\vdots$

$\exists x_1 \in A_1$ such that

$$x_1 < x_2$$

Now

$$x_1 < x_2 < \dots < x_\alpha$$

is a chain.

Take $C = \{x_i\}_{i=1}^\alpha$

□

Example integer $n \geq 1$

$X =$ set of all subsets of $\{1, 2, \dots, n\}$

$x \leq y$ whenever $x \subseteq y$ ($\forall x, y \in X$)

We saw earlier

each max chain in X has size $n+1$

By the theorem we can partition X into $n+1$ antichains.

For $0 \leq i \leq n$ define

$$A_i = \{x \in X \mid |x| = i\}$$

$\{A_i\}_{i=0}^n$ is antichain partition of X .

Given a nonempty finite set X with partial order $\leq$

Suppose we partition X into chains $\{C_i\}_{i=1}^r$

So

$$X = \bigcup_{i=1}^r C_i \quad (\text{disj union})$$

For all antichains A ,

$$A = A \cap X$$

$$= A \cap \left(\bigcup_{i=1}^r C_i \right)$$

$$= \bigcup_{i=1}^r (A \cap C_i)$$

So

$$|A| = \sum_{i=1}^r |A \cap C_i|$$

≤ 1

$$\leq r$$

Therefore

$$\max \{ |A| \mid A \text{ an antichain of } X \} \leq \min \{ r \mid \{C_i\}_{i=1}^r \text{ is chain partition of } X \}$$

(**)

thm (Dilworth)

Equality holds in (**)

pf see text

Example integer $n \geq 1$

$X =$ set of all subsets of $\{1, 2, \dots, n\}$

$x \leq y$ whenever $x \subseteq y$

For $0 \leq i \leq n$ def

$$A_i = \{x \in X \mid |x| = i\}$$

Antichain

$$|A_i| = \binom{n}{i}$$

By Sperner's Thm

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}$$

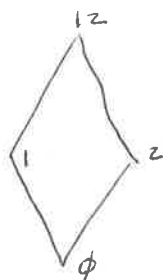
is max'd size of any antichain

$A_{\lfloor \frac{n}{2} \rfloor}$ is max'd antichain

We now partition X into $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ chains

start with $n=2$

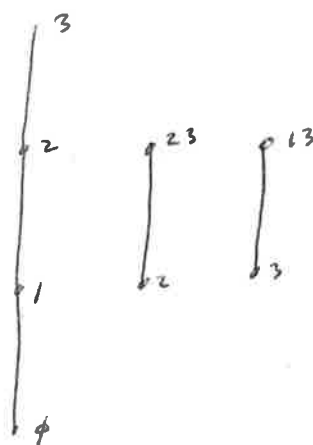
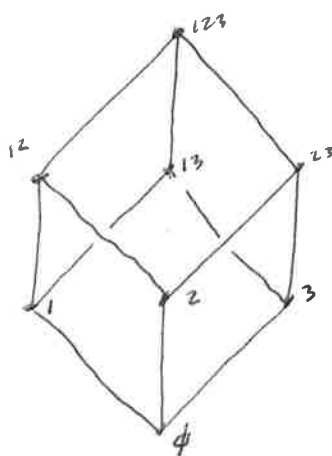
X:



chain decomp



$n=3$



$n=4$ Use previous chain decomp as follows:

Given a chain

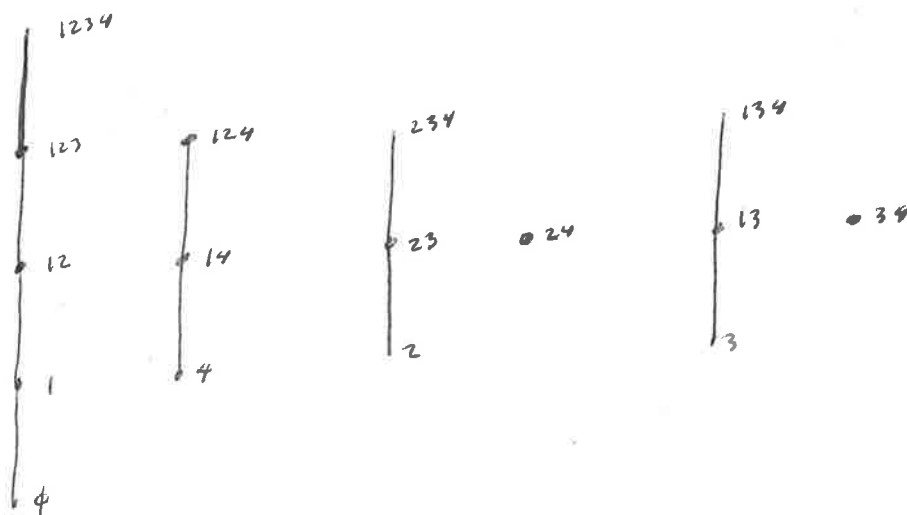
$$x_1 < x_2 < \dots < x_l$$

In previous chain decomp

Create 2 chains if $l \geq 2$ and 1 chain if $l=1$:

chain 1: $x_1 < x_2 < \dots < x_l < x_l \cup \{4\}$

chain 2 (if $l \geq 2$): $x_1 \cup \{4\} < x_2 \cup \{4\} < \dots < x_{l-1} \cup \{4\}$



For $n \geq 5$ chain decomp is obtained from $(n-1)$ -chain decomp
in sim way.

We have now constructed a chain

decomp $\{C_i\}_{i=1}^r$ of X

show
$$r = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

Given chain C_i

write

$C_i: x_1 < x_2 < \dots < x_\ell$

By construction

$$|x_1| + |x_\ell| = n$$

"symmetric"

So

C_i contains unique element of $A_{\lfloor \frac{n}{2} \rfloor}$

so

$$r = |A_{\lfloor \frac{n}{2} \rfloor}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

— 0 —

