Math 846

Lecture 15

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

We continue to discuss a DRG Lec 15

$\Gamma = (X, R)$ with diameter D.

Fix $x \in X$ and write $T = T(x)$, etc

Recall the Krein parameters q^h_{ij}

LEM 65 For $0 \le h, i, j, r \le D$

$$\sum_{\alpha=0}^{D} q^r_{h\alpha} \, q^\alpha_{ij} = \sum_{\beta=0}^{D} q^r_{j\beta} \, q^\beta_{hi}$$

pf Expand each side of

$$A_h^* \left(A_i^* A_j^*\right) = \left(A_h^* A_i^*\right) A_j^*$$

as a linear combination of the dual distance matrices, and compare coefficients

$\square$

LEM 66 With above notation

(i) $q^h_{0j} = \delta_{hj} \qquad (0 \le h, j \le D)$

(ii) $q^h_{i0} = \delta_{hi} \qquad (0 \le h, i \le D)$

(iii) $q^0_{ij} = \delta_{ij} m_i \qquad (0 \le i, j \le D)$

(iv) $\sum_{i=0}^{D} q^h_{ij} = m_j \qquad (0 \le h, j \le D)$

pf (i) By Lem 63,

$$q^h_{0j} = |X|^{-1} m_h^{-1} \langle \underset{\overset{\shortparallel}{I}}{A_0^*} A_j^*, A_h^* \rangle$$

By Lem 62

$$\langle A_j^*, A_h^* \rangle = \delta_{jh} m_h |X|$$

(ii) Sim

(iii) Use Lem 64

(iv)
$$\sum_{i=0}^{D} q^h_{ij} = |X|^{-1} m_h^{-1} \sum_{i=0}^{D} \langle A_i^* A_j^*, A_h^* \rangle \quad (\text{by } L63)$$

$$= m_h^{-1} \langle E_0^* A_j^*, A_h^* \rangle \quad (\text{by } L60(ii))$$

$$= m_h^{-1} m_j \langle E_0^*, A_h^* \rangle \quad (\text{by } L61(i))$$

$$= m_j \quad (\text{by } L61, L62)$$

$\square$

The next result shows how to compute the Krein parameters from the intersection numbers

Thm 67 For a DRG $\Gamma = (X, R)$ of diam D and for $0 \le h, i, j \le D$

$$q^h_{ij} = |X|^{-1} m_i m_j \sum_{r=0}^{D} u_r(\theta_i) u_r(\theta_j) v_r(\theta_h)$$

pf Using Lem 61,

$$\langle A_i^* A_j^*, A_h^* \rangle = \left\langle m_i m_j \sum_{r=0}^{D} u_r(\theta_i) u_r(\theta_j) E_r^*, m_h \sum_{s=0}^{D} u_s(\theta_h) E_s^* \right\rangle$$

$$= m_h m_i m_j \sum_{r=0}^{D} u_r(\theta_i) u_r(\theta_j) u_r(\theta_h) k_r$$

$$= m_h m_i m_j \sum_{r=0}^{D} u_r(\theta_i) u_r(\theta_j) v_r(\theta_h)$$

Now evaluate the above equation using Lem 63 $\square$

Our next general goal is to show that the Krein parameters are nonnegative.

LEM 68 With above notation and for $0 \le h, i, j, r, s, t \le D$

(i) $\langle E_i^* A_j E_h^*, \; E_r^* A_s E_t^* \rangle = \delta_{ir}\delta_{js}\delta_{ht} \, k_h \, p^h_{ij}$

(ii) $\langle E_i A_j^* E_h, \; E_r A_s^* E_t \rangle = \delta_{ir}\delta_{js}\delta_{ht} \, m_h \, q^h_{ij}$

pf (i) $\langle E_i^* A_j E_h^*, \; E_r^* A_s E_t^* \rangle$

$$= \operatorname{tr}\left(E_i^* A_j E_h^* \left(\overline{E_r^* A_s E_t^*}\right)^t \right)$$

$$= \operatorname{tr}\left(E_i^* A_j E_h^* E_t^* A_s E_r^* \right) \qquad \operatorname{tr}(uv) = \operatorname{tr}(vu)$$

$$= \delta_{ir}\delta_{ht} \operatorname{tr}\left(E_i^* A_j E_h^* A_s \right)$$

$$\operatorname{tr}(E_i^* A_j E_h^* A_l) = \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz} (A_l)_{zy}$$

$$= \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} \underbrace{(A_j \circ A_l)_{yz}}_{\delta_{jl} A_j} (E_h^*)_{zz}$$

$$= \delta_{jl} \sum_{y \in X} \sum_{z \in X} (E_i^*)_{yy} (A_j)_{yz} (E_h^*)_{zz}$$

$$= \delta_{jl} \sum_{\substack{y \in \Gamma_i(x) \\ z \in \Gamma_h(x) \\ \partial(y,z) = j}} 1$$

$$= \delta_{jl} \, k_h \, p^h_{ij}$$

(iii) $\langle E_i A_j^* E_h,\ E_r A_s^* E_t \rangle$

$$= \operatorname{tr}\left(E_i A_j^* E_h \left(\overline{E_r A_s^* E_t}\right)^t \right)$$

$$= \operatorname{tr}\left(E_i A_j^* E_h\, E_t A_s^* E_r \right)$$

$$= \delta_{ir}\delta_{ht} \operatorname{tr}\left(E_i A_j^* E_h A_s^* \right)$$

$$\operatorname{tr}\left(E_i A_j^* E_h A_s^* \right) = \sum_{y \in X} \sum_{z \in X} (E_i)_{yz} \underbrace{(A_j^*)_{zz}}_{= |X| (E_j)_{xz}} (E_h)_{zy} \underbrace{(A_s^*)_{yy}}_{= |X| (E_s)_{xy}}$$

$$= |X|^2 \sum_{y \in X} \sum_{z \in X} (E_s)_{xy} (E_i \circ E_h)_{yz} (E_j)_{zx}$$

$$= |X|^2 \left((x,x)\text{-entry of } \quad E_s (E_i \circ E_h) E_j \right)$$

$$= |X| \operatorname{tr}\left(E_s (E_i \circ E_h) E_j \right)$$

$$= |X| \operatorname{tr}\left(\underbrace{(E_i \circ E_h)}_{= |X|^{-1} \sum_{\ell=0}^{D} q_{ih}^{\ell} E_\ell} \underbrace{E_j E_s}_{= \delta_{js} E_j} \right)$$

$$= \delta_{js}\, q_{ih}^{j} \underbrace{\operatorname{tr}(E_j)}_{= m_j}$$

$$= \delta_{js}\, m_h\, q_{ij}^{h}$$

□

Ref to LEM 68, setting

$r = i, \; s = j, \; t = h$

gives

$$\| E_i^* A_j E_h^* \|^2 = k_h \, p_{ij}^h \qquad (0 \le h, i, j \le D)$$

$$\| E_i A_j^* E_h \|^2 = m_h \, q_{ij}^h \qquad (0 \le h, i, j \le D)$$

Thm 69 (Krein condition) We have

$$q_{ij}^h \ge 0 \qquad (0 \le h, i, j \le D)$$

pf Recall $\| B \|^2 \ge 0 \quad \forall B \in \mathrm{Mat}_X(\mathbb{F})$ $\square$

Thm 70 (Triple product relations)

For $0 \le h, i, j \le D$,

(i) $E_i^* A_j E_h^* = 0$ iff $p_{ij}^h = 0$

(ii) $E_i A_j^* E_h = 0$ iff $q_{ij}^h = 0$

pf Recall $\| B \|^2 = 0$ iff $B = 0 \quad \forall B \in \mathrm{Mat}_X(\mathbb{F})$. $\square$

Notation For subspaces Y, Z of $Mat_X(\mathbb{F})$

define $YZ = \mathrm{Span}\{ yz \mid y \in Y, z \in Z \}$

Thm 71 With above notation

(i) the vector space $M^* M M^*$ has an orthogonal basis

$$\{ E_i^* A_j E_h^* \mid 0 \le h, i, j \le D, \ p^h_{ij} \ne 0 \}$$

(ii) the vector space $M M^* M$ has an orthogonal basis

$$\{ E_i A_j^* E_h \mid 0 \le h, i, j \le D, \ q^h_{ij} \ne 0 \}$$

pf Routine using Lem 68, Th 70 $\square$

Next we give an application of the Krein cond. Lec 15
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Until further notice, $\Gamma = (X, R)$ is a DRG with $D = 2$ (strongly-Regular).

We are given a nondegenerate primitive idempotent

$$E = |X|^{-1} \sum_{i=0}^{2} \theta_i^* A_i$$

We assume:

$\exists$ mutually distinct $x, y, z \in X$ s.t. $E\hat{x}$, $E\hat{y}$, $E\hat{z}$ are linearly dependent (*)

We seek the solutions for Γ and E. Let $\theta =$ eigval of Γ for E.

We make some observations

- The matrix

$$\begin{pmatrix} \theta_0^* & \theta_1^* & \theta_1^* \\ \theta_1^* & \theta_0^* & \theta_1^* \\ \theta_1^* & \theta_1^* & \theta_0^* \end{pmatrix} \qquad (**)$$

is singular, since this is a scalar multiple of the inner product matrix for $E\hat{x}, E\hat{y}, E\hat{z}$.

- $\theta_0^* + 2\theta_1^* = 0$

Since
$$\det(**) = (\theta_0^* + 2\theta_1^*)(\theta_0^* - \theta_1^*)^2$$

and $\theta_0^* \neq \theta_1^*$.

- For all mutually adjacent $x, y, z \in X$
$$E\hat{x} + E\hat{y} + E\hat{z} = 0$$

Since $\quad \| E\hat{x} + E\hat{y} + E\hat{z} \|^2 = 3(\theta_0^* + 2\theta_1^*)/|X|^7$
$$= 0$$

- $p_{11}^1 = 1$

Since if $p_{11}^1 \geq 2$ then for mutually adjacent x, y, z $\quad \exists\ z' \in \Gamma(x) \cap \Gamma(y) \quad z \neq z'$

then
$$E\hat{x} + E\hat{y} + E\hat{z} = 0$$
$$E\hat{x} + E\hat{y} + E\hat{z}' = 0$$

So $\quad E\hat{z} = E\hat{z}'$

forcing $\quad z = z' \quad$ since $\quad E$ is nondeg.

- $\theta_1^* + 2\theta_2^* = 0$

Since for 3 mutually adj $x, y, z \in X$ pick $w \in \Gamma(x) \setminus \{y, z\}$. Then

$$\partial(w,y) = 2, \qquad \partial(w,z) = 2 \qquad \text{since } p^1_{11} = 1$$

$$0 = \langle E\hat{w}, E\hat{x} + E\hat{y} + E\hat{z} \rangle$$

$$= |X|^{-1}(\theta_1^* + 2\theta_2^*).$$

- $\exists$ integer $r \geq 2$ st

$$c_2 = r, \qquad k = 2r, \qquad b_1 = 2(r-1)$$

To see this, evaluate

$$c_i \theta_{i-1}^* + a_i \theta_i^* + b_i \theta_{i+1}^* = \theta \theta_i^* \qquad i = 0, 1, 2$$

and use

$$\theta_0^* + 2\theta_1^* = 0, \qquad \theta_1^* + 2\theta_2^* = 0, \qquad a_1 = 1$$

- the eigenvalues of Γ are

$$\theta_0 = 2r \qquad \theta_1 = 1 \qquad \theta_2 = -r$$

and $\theta = \theta_2$

- For Γ the table of cosines is

	A_0	A_1	A_2
E_0	1	1	1
E_1	1	$\frac{1}{2r}$	$\frac{-1}{2(r-1)}$
E_2	1	$-\frac{1}{2}$	$\frac{1}{4}$

$(E = E_2)$

- the eigenvalue multiplicities are

$$m_0 = 1, \qquad m_1 = \frac{6r(r-1)}{r+1} \qquad m_2 = \frac{4(2r-1)}{r+1}$$

- $r \in \{2, 3, 5\}$

Indeed $m_2 = 8 - \frac{12}{r+1}$

So $r+1$ divides 12

Also using $q_{22}^2 \geq 0$ we find

$$r \leq 5$$

- For each $r \in \{2, 3, 5\}$ we recognize Γ:

$r = 2$: $|X| = 9$ $k = 4$, $b_1 = 2$

$$\mathrm{Spec}(\Gamma) = \begin{pmatrix} 4 & 1 & -2 \\ 1 & 4 & 4 \end{pmatrix}$$

$$\Gamma = K_3 \times K_3 = H(3, 2)$$

$r=3$: $|X| = 15$, $k=6$, $b_1 = 4$

$$\operatorname{spec}(\Gamma) = \begin{pmatrix} 6 & 1 & -3 \\ 1 & 9 & 5 \end{pmatrix}$$

Γ is complement of Johnson graph $J(6,2)$

$r=5$: $|X| = 27$, $k=10$, $b_1 = 8$

$$\operatorname{spec}(\Gamma) = \begin{pmatrix} 10 & 1 & -5 \\ 1 & 20 & 6 \end{pmatrix}$$

Γ is the complement of the Schläfli graph.