

Combinatorics 475

Lecture 14

Sperner TheoremMotivation

Given any poset $X, \leq$ $|X| < \infty$

A chain in $X, \leq$ is a subset $C \subseteq X$ such that any two elements of C are comparable

The elements $x_1, x_2, \dots, x_t$ of a chain C can be ordered s.t.

$$x_1 < x_2 < \dots < x_t$$

A chain C is maximal whenever C is not properly

contained in another chain.

An antichain in $X, \leq$ is a subset $A \subseteq X$ such that any 2 distinct elements of A are incomparable

Ex

Fix integer $n \geq 1$

X : all subsets of $\{1, 2, \dots, n\}$

$x \leq y$ whenever $x \subseteq y$

$\forall x, y \in X$

so $|X| = 2^n$

Describe max'l chains

Max'l chain C has form $C = \{x_i\}_{i=0}^n$ such that

$$|x_i| = i \quad 0 \leq i \leq n$$

and

$$x_0 \subseteq x_1 \subseteq x_2 \subseteq \dots \subseteq x_n$$

there is a bijection between the
set of max'l chains of $X_1 \leq$ and the perms of $\{1, 2, \dots, n\}$.

Given a perm $a_1, a_2, \dots, a_n$ of $\{1, 2, \dots, n\}$

Define a subset $X_i \subseteq \{1, 2, \dots, n\}$ by

$$X_i = \{a_1, a_2, \dots, a_i\}$$

$$0 \leq i \leq n$$

then

$$\{X_i\}_{i=0}^n \text{ is a max'l chain in } X_1 \leq.$$

Conversely, given max'l chain C of $X_1 \leq$

$$\text{write } C = \{X_i\}_{i=0}^n \text{ with } |X_i| = i \text{ for } 0 \leq i \leq n$$

note $X_0 = \emptyset$
def

$$a_1 = \text{unique elem } X_1$$

$$a_2 = \text{ " } X_2 \setminus X_1$$

$$a_3 = \text{ " } X_3 \setminus X_2$$

$$\vdots$$

$$a_n = \text{ " } X_n \setminus X_{n-1}$$

then $a_1, a_2, \dots, a_n$ is perm of $\{1, 2, \dots, n\}$

In summary

$$\left\{ \begin{array}{c} \text{perms} \\ \text{of } \{1, 2, \dots, n\} \end{array} \right\} \xleftrightarrow{\text{bij}} \left\{ \begin{array}{c} \text{max'l} \\ \text{chains} \\ \text{in } X_1 \leq \end{array} \right\}$$

Σ_0 $X_1 \leq$ has $n!$ max'l chains.

Now consider antichains of $X, \leq$

For $0 \leq k \leq n$ define

$$A_k = \{ x \in X \mid |x| = k \}$$

So $|A_k| = \binom{n}{k}$

$$X = \bigcup_{k=0}^n A_k \quad \text{disjoint union}$$

Obs each A_k is an antichain of $X, \leq$

Take $k = \lfloor \frac{n}{2} \rfloor$:

$X, \leq$ has an antichain with cardinality $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

Thm (Sperner) For the above poset $X, \leq$

each antichain has cardinality at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$

pf Given antichain A of $X, \leq$

LEC 14

4

Count # of ordered pairs (a, C) such that

$a \in A$ and C is max'l chain containing a

Count in 2 ways

Count 1

$n!$ choices for C

Given C at most one choice for a

$$\# \leq n!$$

Count 2

For $a \in A$

write $|a| = k$

There are $k! (n-k)!$

max'l chains containing a

For $0 \leq k \leq n$ let

$d_k =$ number of elements of A of size k

so

$$|A| = \sum_{k=0}^n d_k$$

so

$$\# = \sum_{k=0}^n d_k k! (n-k)!$$

so

$$n! \geq \sum_{k=0}^n \alpha_k k! (n-k)!$$

$$1 \geq \sum_{k=0}^n \alpha_k \frac{k! (n-k)!}{n!}$$

$$= \sum_{k=0}^n \frac{\alpha_k}{\binom{n}{k}}$$

$$\geq \sum_{k=0}^n \frac{\alpha_k}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

$$= \frac{|A|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$$

so

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq |A|$$

□

5.4 Multinomial Theorem

Recall binomial theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Now consider $t \geq 2$ variables

$$x_1, x_2, \dots, x_t$$

Expand

$$(x_1 + x_2 + \dots + x_t)^n$$

$$n = 0, 1, 2, \dots$$

Sol

$$(x_1 + \dots + x_t)^n = (x_1 + \dots + x_t)(x_1 + \dots + x_t) \dots$$

(n factors)

$$= \sum a_1 a_2 \dots a_n$$

sum is over all sequences $a_1, a_2, \dots, a_n$ such that

$$a_i \in \{x_1, x_2, \dots, x_t\} \text{ for } 1 \leq i \leq n$$

Each summand $a_1 a_2 \dots a_n$ contributes

$$x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$$

where

$$n_i = \left| \{ j \mid a_j = x_i \} \right| \quad 1 \leq i \leq t$$

Note $n = n_1 + n_2 + \dots + n_t$

Given integer $n_1, n_2, \dots, n_t \geq 0$ s.t. $n = n_1 + n_2 + \dots + n_t$

the number of summands $a_1 a_2 \dots a_n$ that contribute *

= # ways to fill n blanks

— — — — —

with

n_1 copies x_1
 n_2 " x_2
 $\dots$
 n_t " x_t

= # perms of multiset

$$\{ n_1 \cdot x_1, n_2 \cdot x_2, \dots, n_t \cdot x_t \}$$

$$= \frac{n!}{n_1! n_2! \dots n_t!}$$

so

thm (Multinomial thm)

$$(x_1 + x_2 + \dots + x_t)^n = \sum \frac{n!}{n_1! n_2! \dots n_t!} x_1^{n_1} x_2^{n_2} \dots x_t^{n_t}$$

where sum is over all non negative integral solutions $n_1, n_2, \dots, n_t$ to $n_1 + n_2 + \dots + n_t = n$

Def We abbrev

$$\binom{n}{n_1, n_2, \dots, n_t} = \frac{n!}{n_1! n_2! \dots n_t!}$$

$$n = n_1 + n_2 + \dots + n_t$$

$$n_1, n_2, \dots, n_t \geq 0$$

"multinomial coefficient"

ex take $t=2$

$$\binom{n}{n_1, n_2} = \frac{n!}{n_1! n_2!}$$

$$n_1 + n_2 = n$$

$$n_1 \geq 0, n_2 \geq 0$$

Write $k = n_1$
 $n - k = n_2$

$$\binom{n}{k, n-k} = \frac{n!}{k! (n-k)!} = \underbrace{\binom{n}{k}}_{\text{usual binom coeffs}} = \binom{n}{n-k}$$

Recall Pascal's formula

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

We now display a multinomial analog.

thm. Given positive integers $n_1, n_2, \dots, n_t$ and write

$$n = n_1 + n_2 + \dots + n_t$$

Then

$$\begin{aligned} \binom{n}{n_1, n_2, \dots, n_t} &= \binom{n-1}{n_1-1, n_2, \dots, n_t} + \binom{n-1}{n_1, n_2-1, n_3, \dots, n_t} + \dots \\ &\quad + \binom{n-1}{n_1, n_2, \dots, n_t-1} \\ &= \sum_{i=1}^t \binom{n-1}{n_1, \dots, n_{i-1}, n_i-1, n_{i+1}, \dots, n_t} \end{aligned}$$

pf1 (Algebraic) evaluate each term using def of multinomial coeff

pf2 (combinatorial)

Consider multiset

$$\{n_1 \cdot x_1, n_2 \cdot x_2, \dots, n_t \cdot x_t\}$$

let P = set of permutations of this multiset

$|P_i| = \# \text{ perms of multiset}$

$$\left\{ n_1 \cdot x_1, n_2 \cdot x_2, \dots, n_m \cdot x_m, (n_i - 1) \cdot x_i, n_{i+1} \cdot x_{i+1}, \dots, n_t \cdot x_t \right\}$$

$$= \binom{n-1}{n_1 \dots n_{i-1} \quad n_i-1 \quad n_{i+1} \dots n_t}$$

Result follows. □

