Math 846

Lecture 14

Next topic: the Krein parameters of a DRG

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ Given DRG $\Gamma = (X, \mathcal{R})$ diam D.

For $B, C \in \mathrm{Mat}_X(\mathbb{F})$ define

$$B \circ C \in \mathrm{Mat}_X(\mathbb{F})$$

by
$$(B \circ C)_{xy} = B_{xy} C_{xy} \qquad \forall x, y \in X$$

"entry-wise multiplication"

Obs
$$A_i \circ A_j = \delta_{ij} A_i \qquad (0 \le i, j \le D)$$

So adjacency algebra M is closed under $\circ$

So $\exists\ q^h_{ij} \in \mathbb{F}$ $(0 \le h, i, j \le D)$ st

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q^h_{ij} E_h \qquad (0 \le i, j \le D)$$

By constr
$$q^h_{ij} = q^h_{ji} \qquad (0 \le h, i, j \le D)$$

the primitive idempotents are real so $q^h_{ij} \in \mathbb{R}$ $(0 \le h, i, j \le D)$

We will show
$$q^h_{ij} \ge 0 \qquad (0 \le h, i, j \le D)$$

The q^h_{ij} are called the <u>Krein parameters</u> of Γ.

To avoid working with entrywise mult $\circ$, we bring in the dual adj algebra.

Until further notice fix $x \in X$

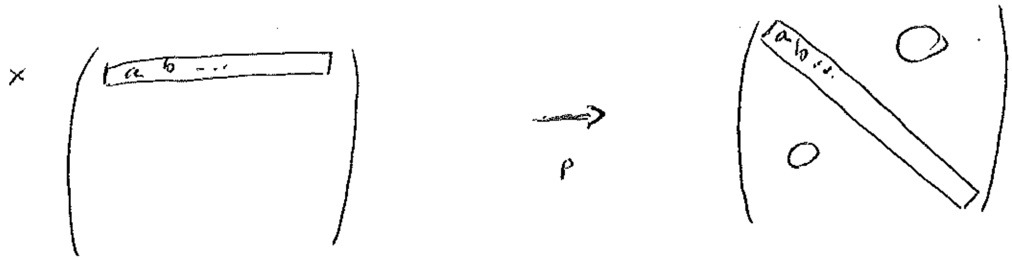
Recall for $0 \le i \le D$ the diagonal matrix $E_i^* = E_i^*(x) \in Mat_X(\mathbb{F})$ has (y,y)-entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } y \in \Gamma_i(x) \\ 0 & \text{if } y \notin \Gamma_i(x) \end{cases} \qquad y \in X$$

Recall $\{E_i^*\}_{i=0}^D$ is basis for algebra $M^* = M^*(x)$

Def 52 With above notation, $\forall\ B \in Mat_X(\mathbb{F})$ let B^ρ denote the diagonal matrix in $Mat_X(\mathbb{F})$ with (y,y)-entry

$$(B^\rho)_{yy} = B_{xy} \qquad y \in X$$



B $\qquad\qquad$ B^ρ

LEM 53 With above notation,

$$(B \circ C)^\rho = B^\rho C^\rho \qquad \forall B, C \in Mat_X(\mathbb{F})$$

pf clear $\square$

LEM 54 With above notation,

(i) $(A_i)^\rho = E_i^*$ $(0 \le i \le D)$

(ii) $I^\rho = E_0^*$.

(iii) $J^\rho = I$

pf (i) $\forall y \in X$

$$(A_i^\rho)_{yy} = (A_i)_{xy} = (E_i^*)_{yy}$$

(ii) Set $i=0$ in (i)

(iii) By def of ρ. $\square$

LEM 55 With above notation the restriction

$$\rho|_M : M \to M^*$$

is an isomorphism of vector spaces

[Caution: not an iso of algebras]

pf ρ sends the basis $\{A_i\}_{i=0}^{D}$ of M to the basis $\{E_i^*\}_{i=0}^{D}$ of M^*. $\square$

LEM 56 With above notation, $\forall\ B, C \in M$ we have

$$\langle B^\rho, C^\rho \rangle = |X|^{-1} \langle B, C \rangle.$$

Moreover this quantity equals the (x,x)-entry of $B\overline{C}^t$.

pf

$$\begin{aligned}
\langle B^\rho, C^\rho \rangle &= tr\left(B^\rho \, \overline{(C^\rho)}^t \right) \\
&= tr\left(B^\rho \, \overline{C^\rho} \right) \\
&= \sum_{y \in X} (B^\rho)_{yy} \, \overline{C^\rho}_{yy} \\
&= \sum_{y \in X} B_{xy} \, \overline{C}_{xy} \\
&= (B \overline{C}^t)_{xx}
\end{aligned}$$

Also

$$\begin{aligned}
|X|^{-1} \langle B, C \rangle &= |X|^{-1} \, tr(B \overline{C}^t) \\
&= |X|^{-1} \sum_{y \in X} (B \overline{C}^t)_{yy} \\
&= (B \overline{C}^t)_{xx}
\end{aligned}$$

since diagonal entries of $B\overline{C}^t$ are all equal

$\square$

We mention a useful fact abt the map ρ.

LEM 57 With above notation,

$$E_0 E_0^* B = E_0 B^\rho \qquad \forall B \in M$$

pf Recall $E_0 = |X|^{-1} J$ and

$$E_0^* = \begin{array}{cc} & x \\ x & \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & O \end{array}\right) \end{array}$$

$\forall y, z \in X$

$$(E_0 E_0^* B)_{yz} = \underbrace{(E_0)_{yx}}_{= |X|^{-1}} \underbrace{(E_0^*)_{xx}}_{= 1} B_{xz}$$

$$= |X|^{-1} B_{xz}$$

Also since B^ρ is diagonal

$$(E_0 B^\rho)_{yz} = (E_0)_{yy} (B^\rho)_{zz}$$

$$= |X|^{-1} B_{xz}$$

$\square$

The situation so far:

$$\begin{tikzcd}
A_i \arrow[r] & E_i^* \\
E_i \arrow[r] & ? \\
M \arrow[r, "\rho"'] & M^*
\end{tikzcd}$$

DEF 58 With above notation, for $0 \le i \le D$ define $A_i^* = A_i^*(x)$ by

$$A_i^* = |X| \, (E_i)^\rho$$

so A_i^* is diagonal with (y,y)-entry

$$(A_i^*)_{yy} = |X| \, (E_i)_{xy}$$
$$= m_i \, u_h(\theta_i) \qquad h = \partial(x,y)$$

Call A_i^* the <u>ith dual distance matrix of Γ with respect to x</u>

LEM 59 With above notation,

(i) $\{A_i^*\}_{i=0}^{D}$ is a basis for M^*

(ii) For $0 \le i \le D$ and $y \in X$,

$$A_i^* \hat{y} = m_i u_h(\theta_i) \hat{y} \qquad h = \partial(x,y)$$

pf (i) By Lem 55 and since $\{E_i\}_{i=0}^{D}$ is a basis for M.

(ii) By Def 58. $\square$

LEM 60 With above notation,

(i) $A_0^* = I$

(ii) $\sum_{i=0}^{D} A_i^* = |X| \, E_0^*$.

(iii) $\overline{A_i^*} = A_i^*$ $(0 \le i \le D)$

(iv) $(A_i^*)^t = A_i^*$ $(0 \le i \le D)$

(v) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \le i, j \le D)$

pf (i) $A_0^* = |X| \, E_0^\rho$ and $E_0 = |X|^{-1} J$

(ii) Apply ρ to
$$\sum_{i=0}^{D} E_i = J$$

(iii), (iv) clear

(v) Apply ρ to
$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h$$

$\square$

LEM 61 With above notation

(i) $$A_j^* = m_j \sum_{i=0}^{D} u_i(\theta_j)\, E_i^* \qquad (0 \le j \le D)$$

(ii) $$E_j^* = |X|^{-1} \sum_{i=0}^{D} v_j(\theta_i)\, A_i^* \qquad (0 \le j \le D)$$

pf Apply ρ to the equations in LEM 20. $\square$

LEM 62 With above notation.

(i) $$\langle E_i^*, E_j^* \rangle = \delta_{ij} k_i \qquad (0 \le i,j \le D)$$

(ii) $$\langle A_i^*, A_j^* \rangle = \delta_{ij} m_i |X| \qquad (0 \le i,j \le D)$$

pf Combine lemmas 34, 56. $\square$

Algebraically, the p_{ij}^h and q_{ij}^h are very similar, as the next few results illustrate.

LEM 63 With the above notation,

for $0 \le h, i, j \le D$

$$q_{ij}^h = |X|^{-1} m_h^{-1} \langle A_i^* A_j^*, A_h^* \rangle$$
$$= |X|^{-1} m_h^{-1} \langle A_h^*, A_i^* A_j^* \rangle$$

pf Expand $A_i^* A_j^*$ using

$$A_i^* A_j^* = \sum_{l=0}^{D} q_{ij}^l A_l^*$$

and use Lem 62 $\square$

LEM 64 With above notation,

$$m_h q^h_{ij} = m_i q^i_{jh} = m_j q^j_{hi} \qquad (0 \le h, i, j \le D)$$

pf To obtain $m_h q^h_{ij} = m_i q^i_{hj}$ observe

$$
\begin{aligned}
|X| \, m_h q^h_{ij} &= \langle A_i^* A_j^*, A_h^* \rangle \\
&= \langle A_i^*, A_h^* (\overline{A_j^*})^t \rangle \\
&= \langle A_i^*, A_h^* A_j^* \rangle \\
&= |X| \, m_i q^i_{jh}
\end{aligned}
$$

$\square$

The rest is similarly obtained