

Combinatorics 475

Lecture 13

Show
$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$$

Consider the set

$$\{\pm 1, \pm 2, \dots, \pm n\}$$

$2n$ elements

Let $S =$ set of all n -element subsets of $\{\pm 1, \pm 2, \dots, \pm n\}$

$$|S| = \binom{2n}{n}$$

For $0 \leq k \leq n$ define

$$S_k = \{x \in S \mid x \text{ contains exactly } k \text{ pos elements}\}$$

$$S = \bigcup_{k=0}^n S_k \quad (\text{disjoint union})$$

$$|S| = \sum_{k=0}^n |S_k|$$

For $0 \leq k \leq n$ to find $|S_k|$ construct $x \in S_k$ via 2-stage exp:

stage	to do	# choices
1	select pos elements of x	$\binom{n}{k}$
2	select neg els of x	$\binom{n}{n-k} (= \binom{n}{k})$

$$|S_k| = \binom{n}{k}^2$$

Result follows. □

We now extend the definition of the binomial coeffs.

Given a variable z

Given integer k

We define

$$\binom{z}{k}$$

as follows:

Case $k \geq 0$

$$\binom{z}{k} = \frac{z(z-1)(z-2) \cdots (z-k+1)}{k!}$$

[polynomial in z with degree k]

Case $k < 0$

$$\binom{z}{k} = 0$$

We can let $z \rightarrow$ any real number

ex

$$\binom{5/2}{3} = \frac{5/2 \cdot 3/2 \cdot 1/2}{3 \cdot 2 \cdot 1}$$

$$= 5/16$$

Pascal's formula holds for gen z, k :

$$\binom{z}{k} = \binom{z-1}{k-1} + \binom{z-1}{k}$$

(or)

Last time we saw, for integers $0 \leq k \leq n$

$$\binom{k}{k} + \binom{k+1}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

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Next goal: find analog that holds after $n \rightarrow z$

Modify * using

$$\binom{n}{k} = \binom{n}{n-k}$$

Get

$$\binom{k}{0} + \binom{k+1}{1} + \dots + \binom{n-1}{n-k+1} + \binom{n}{n-k} = \binom{n+1}{n-k+1}$$

Replace $k \rightarrow n-k$

$$\binom{n-k}{0} + \binom{n-k+1}{1} + \dots + \binom{n-1}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Replace $n \rightarrow z$ even?

$$\sum_{i=0}^k \binom{z-i}{k-i} = \binom{z+1}{k}$$

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thm $**$ holds for a variable z
and any integer $k \geq 0$

p.f. Repeatedly apply Pascal formula to RHS □

ex take $k=3$

$$\begin{array}{ccccccc}
 \binom{z}{3} & + & \binom{z-1}{2} & + & \binom{z-2}{1} & + & \binom{z-3}{0} & = & \binom{z+1}{3} \\
 // & & // & & // & & // & & // \\
 \frac{z(z-1)(z-2)}{6} & & \frac{(z-1)(z-2)}{2} & & z-2 & & 1 & & \frac{(z+1)z(z-1)}{6}
 \end{array}$$

5.3 Unimodality of Binomial Coef

Consider any row of Pascal's triangle

ex row 8

$$1 \leq 8 \leq 28 \leq 56 \leq 70 \geq 56 \geq 28 \geq 8 \geq 1$$

Sequence is unimodal

A finite sequence of real numbers

$$a_0, a_1, \dots, a_n$$

is unimodal whenever $\exists t$ $0 \leq t \leq n$ such that

$$a_0 \leq a_1 \leq \dots \leq a_t \geq a_{t+1} \geq \dots \geq a_n$$

ex the sequence

$$1, 4, 5, 9, 9, 9, 6, 3$$

is unimodal

Next goal: Show that each row in Pascal triangle is unimodal

Thm Given integer $n \geq 0$

(i) For n even

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \dots < \binom{n}{n/2}$$

$$\binom{n}{n/2} > \dots > \binom{n}{n-1} > \binom{n}{n}$$

(ii) For n odd

$$\binom{n}{0} < \binom{n}{1} < \binom{n}{2} < \dots < \binom{n}{\frac{n-1}{2}}$$

$$\binom{n}{\frac{n+1}{2}} = \binom{n}{\frac{n-1}{2}}$$

$$\binom{n}{\frac{n+1}{2}} > \dots > \binom{n}{n-1} > \binom{n}{n}$$

In particular row n of Pascal triangle is unimodal.

Pf For $1 \leq k \leq n$ compare

$$\binom{n}{k-1}, \quad \binom{n}{k}$$

$$\frac{\binom{n}{k}}{\binom{n}{k+1}} \begin{matrix} ? \\ > \\ = \\ < \end{matrix} 1$$

Find

$$\frac{\binom{n}{k}}{\binom{n}{k+1}} - 1 \begin{matrix} ? \\ > \\ = \\ < \end{matrix} 0$$

$$\binom{n}{k} = \frac{n(n-1) \cdots (n-k+2)(n-k+1)}{k!}$$

$$\binom{n}{k+1} = \frac{n(n-1) \cdots (n-k+2)}{(k+1)!}$$

$$\frac{\binom{n}{k}}{\binom{n}{k+1}} = \frac{n-k+1}{k+1}$$

$$\frac{\binom{n}{k}}{\binom{n}{k+1}} - 1 = \frac{n-k+1}{k+1} - 1 = \frac{n-k+1 - (k+1)}{k+1} = \frac{n-2k}{k+1}$$

Find sign of

$n - 2k + 1$

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For n even

k	sign of $n - 2k + 1$
$1 \leq k \leq \frac{n}{2}$	+
$\frac{n}{2} < k \leq n$	-

For n odd

k	sign of $n - 2k + 1$
$1 \leq k \leq \frac{n-1}{2}$	+
$k = \frac{n+1}{2}$	0
$\frac{n+1}{2} < k \leq n$	-

□

Notation

For a real number r

def

$$\lfloor r \rfloor = \text{greatest integer at most } r$$

$$= \max \{ n \mid n \in \mathbb{Z}, n \leq r \}$$

"floor of r "

$$\lceil r \rceil = \text{least integer at least } r$$

$$= \min \{ n \mid n \in \mathbb{Z}, n \geq r \}$$

"ceiling of r "

ex For an integer n

$$\left\lfloor \frac{n}{2} \right\rfloor = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$$

$$\left\lceil \frac{n}{2} \right\rceil = \begin{cases} \frac{n}{2} & \text{if } n \text{ even} \\ \frac{n+1}{2} & \text{if } n \text{ odd} \end{cases}$$

We now consider the previous thm using the floor/ceiling notation

Cor For an integer $n \geq 0$

$$\binom{n}{\lfloor \frac{n}{2} \rfloor} = \binom{n}{\lceil \frac{n}{2} \rceil}$$

Moreover this common value is the largest among

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \dots, \binom{n}{n}$$

Pf Routine using prev thm.

□

