Math 846

Lecture 13

Given a representation $(\rho, H)$ of a DRG $\Gamma = (X, R)$

the corresponding Gram matrix is the matrix of

inner products

$$E = \Big( \langle \rho(x), \rho(y) \rangle \Big)_{x,y \in X}$$

$$\in \mathrm{Mat}_X(\mathbb{F})$$

Representations $(\rho, H)$ and $(\rho', H')$ of $\Gamma$

are called equivalent whenever their

Gram matrices $E$, $E'$ satisfy

$$E' \in \mathrm{Span}(E)$$

Usually we do not distinguish between equivalent reps of $\Gamma$.

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

LEM 48 For a DRG $\Gamma = (X, \mathcal{R})$ with diameter $D$, and a primitive idempotent $E_j$ of $\Gamma$ Lec 13 2

(i) $(\rho, H)$ is a representation of $\Gamma$, where

$H = E_j V$ with inner product inherited from $V$

and

$$\rho : \begin{array}{ccc} X & \to & H \\ x & \to & E_j \hat{x} \end{array}$$

(ii) For this rep the Gram matrix is $E_j$

(iii) For $0 \le i \le D$ and $x \in X$

$$\sum_{y \in \Gamma_i(x)} \rho(y) = v_i(\theta_j)\, \rho(x) \qquad (*)$$

(iv) the rep is nondegenerate iff $u_i(\theta_j) \neq 1$ $(1 \le i \le D)$

pf (i) – (iii) Check axioms R1–R3 in DEF 46.

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R1 : $\operatorname{Span}(\rho(x) \mid x \in X) = \text{colspace of } E_j$
$$= E_j V$$
$$= H$$

R2 Gram matrix is $E_j$ by LEM 40 (i)

R3 To verify $(*)$ consider col $x$ in

$$E_j A_i = v_i(\theta_j) E_j$$

Note R3 is just $(*)$ with $i = 1$

(iv) By Cor 43.

thm 49 Given a DRG $\Gamma = (X, \mathcal{R})$ diam $D$.

Given a representation $(\rho, H)$ of $\Gamma$.

Then this rep is equivalent to a rep of $\Gamma$

from LEM 48, for a unique $\theta$.

pf Consider Gram matrix

$$E = \left( \langle \rho(x), \rho(y) \rangle \right)_{x, y \in X}.$$

show $E$ is a nonzero scalar multiple of a

primitive idempotent of $\Gamma$.

Obs $E \neq 0$ by R1

By R2 $\exists\ \alpha_i \in \mathbb{F}$ $(0 \le i \le D)$ st

$$E = \sum_{i=0}^{D} \alpha_i A_i$$

So $E \in M$

Pick $x \in X$. By R3 $\exists\ \theta \in \mathbb{F}$ st

$$\sum_{y \in \Gamma(x)} \rho(y) = \theta \rho(x) \qquad \star$$

In $(*)$ take $\langle\, ,\, \rangle$ of each side with $\rho(x)$
to find $\theta$ is indep of $x$.

Now $(*)$ implies

$$EA = \theta E$$

So $\theta$ is an eigenvalue of $\Gamma$ and

$$E \in \mathrm{Span}(E_j) \qquad \text{where } \theta = \theta_j \qquad \square$$

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Given a DRG $\Gamma = (X, R)$ of diameter $D$

Given a primitive idempotent $E_j$ of $\Gamma$

We call $E_j$ <u>nondegenerate</u> whenever

$$u_i(\theta_j) \neq 1 \qquad (1 \le i \le D)$$

[ ie whenever the corresp rep in LEM 48 is nondegenerate ]

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

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Recall the Hamming graph $H(D,N)$

has intersection numbers

$$c_i = i, \qquad b_i = (N-1)(D-i) \qquad (0 \le i \le D)$$

Obs $|X| = N^D$. $\qquad k_i = (N-1)^i \binom{D}{i} \qquad (0 \le i \le D)$.

Next goal: explain how the corresponding polynomials $\{u_i\}_{i=0}^D$ are Krawtchouk polynomials

LEM 50 The Hamming graph $H(D,N)$

has spectrum

$$\begin{pmatrix} \theta_0, \theta_1, \ldots & \theta_D \\ m_0, m_1 \ldots & m_D \end{pmatrix}$$

where

$$\theta_i = (D-i)(N-1) - i \qquad 0 \le i \le D$$

$$m_i = (N-1)^i \binom{D}{i}$$

pf $H(D,N)$ is the Cartesian product of complete graphs

$$H(D,N) = K_N \times K_N \times \cdots \times K_N \qquad (D \text{ copies})$$

Each $K_N$ has spec $\begin{pmatrix} N-1, & -1 \\ 1 & N-1 \end{pmatrix}$

Now use Lem 14 in chapter 1 $\square$

Notation

For an integer $i \geq 0$ and $a \in \mathbb{F}$

$$(a)_i = \underbrace{a(a+1)(a+2)\cdots(a+i-1)}_{i \text{ factors}}$$

We interp
$$(a)_0 = 1$$

Obs for integers $i, j \geq 0$

$$(-j)_i = \begin{cases} \neq 0 & \text{if } 0 \leq i \leq j \\ 0 & \text{if } i \geq j+1 \end{cases}$$

For integers $r, s \geq 0$ and scalars

$\alpha_1, \alpha_2, \ldots, \alpha_r \qquad \beta_1, \beta_2, \ldots, \beta_s \quad \in \mathbb{F}$

the corresp <u>hypergeometric series</u> in a variable $z$ is

$${}_rF_s\left[\begin{matrix}\alpha_1\ \alpha_2 \cdots \alpha_r \\ \beta_1\ \beta_2 \cdots \beta_s\end{matrix} \,\middle|\, z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_r)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_s)_n}\frac{z^n}{n!}$$

If at least one of $\alpha_1, \alpha_2, \ldots \alpha_r$ is an integer $\leq 0$ then series has finitely many nonzero terms. In this course we consider this situation only.

Fix an integer $D \geq 0$ and $p \in \mathbb{F} \setminus \{0,1\}$

For $0 \leq i \leq D$ define $K_i = K_i(\lambda, p, D) \in \mathbb{F}[\lambda]$

by
$$K_i = {}_2F_1\left[\begin{matrix} -i, & -\lambda \\ & -D \end{matrix} \;\middle|\; \frac{1}{p}\right]$$

So
$$K_0 = 1$$

$$K_1 = 1 - \frac{\lambda}{D}\frac{1}{p}$$

$$K_2 = 1 - \frac{2\lambda}{D}\frac{1}{p} + \frac{\lambda(\lambda-1)}{D(D-1)}\frac{1}{p^2}$$

$$\vdots$$

Call $\{K_i\}_{i=0}^{D}$ the <u>Krawtchouk polynomials</u>

with parameters $D, p$.

One checks that $\{K_i\}_{i=0}^{D}$ satisfy the 3-term recurrence

$$-\lambda K_i = p(D-i)K_{i+1} - \big(p(D-i) + i(1-p)\big)K_i + i(1-p)K_{i-1}$$
$$(0 \le i \le D)$$

Comparing this with our 3-term rec for $\{u_i\}_{i=0}^{D}$ we find

LEM 51 For the Hamming graph $H(D,N)$

$$u_i(\theta_j) = K_i(j) \qquad (0 \le i, j \le D)$$

where
$$K_i = K_i(\lambda, p, D) \qquad p = 1 - \frac{1}{N}$$
$\square$

As an aside, we mention some facts about Krawtchouk polynomials for arb $D$, $p$.

- The (row) orthogonality is

$$\sum_{r=0}^{D} K_i(r) K_j(r) \binom{D}{r} p^r (1-p)^{D-r} = \delta_{ij} \binom{D}{i}^{-1} \left(\frac{1-p}{p}\right)^i$$

$(0 \le i, j \le D)$

- Duality relation:

$$K_i(j) = K_j(i) \qquad (0 \le i, j \le D)$$

- Difference equation

$$-i K_i(j) = p(D-j) K_i(j+1) - \big(p(D-j) + j(1-p)\big) K_i(j) + j(1-p) K_i(j-1) \qquad (0 \le i, j \le D)$$

- "Forward shift operator"

$$K_i(j+1, p, D) - K_i(j, p, D) = \frac{-i}{Dp} K_{i-1}(j, p, D-1)$$

- "Backward shift operator"

$$(D+1-\theta)K_i(\theta,p,D) - \theta\frac{1-p}{p}K_i(\theta-1,p,D) = (D+1)K_{i+1}(\theta,p,D+1)$$

- Generating function: for $0 \le \theta \le D$,

$$\left(1-\frac{1-p}{p}t\right)^{\theta}(1+t)^{D-\theta} = \sum_{i=0}^{D}\binom{D}{i}K_i(\theta,p,D)\,t^i$$

$t$ = indeterminate

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The Krawtchouk polynomials are members of a very general class of orthogonal polynomials called the <u>Askey scheme</u>. We will discuss the Askey scheme a bit later.