

Combinatorics 475

Lecture 12

Ch 5 Binomial Coefficients

Binomial coeffs came up in Ch 2. We now consider them
in depth. Recall for integers $0 \leq k \leq n$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$= \# \text{ } k\text{-subsets of } \{1, 2, \dots, n\}$$

We saw

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad 1 \leq k \leq n$$

"Pascal Formula"

We now find more identities for binom coeffs

Ex Consider a tetrahedral pyramid of stacked cannon balls of height n . How many balls?

Sol

For $n=3$ top view



Floor 1 2 3

$$\text{ans} = 1 + 3 + 6 = 10$$

Gen n :

Floor 1



#balls on floor 1 is

$$1 + 2 + 3 + \dots + n = \binom{n+1}{2}$$

↑
by ind

Similar formula for each floor.

total #balls =

$$\binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \dots + \binom{4}{2} + \binom{3}{2} + \binom{2}{2}$$

Now simplify using Pascal formula:

$$\begin{array}{c}
 \binom{n+1}{2} + \binom{n}{2} + \binom{n-1}{2} + \dots + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} \\
 \underbrace{\hspace{10em}}_{\binom{5}{3}} \\
 \vdots \\
 \underbrace{\hspace{10em}}_{\binom{n+1}{3}} \\
 \underbrace{\hspace{10em}}_{\binom{n+2}{3}}
 \end{array}$$

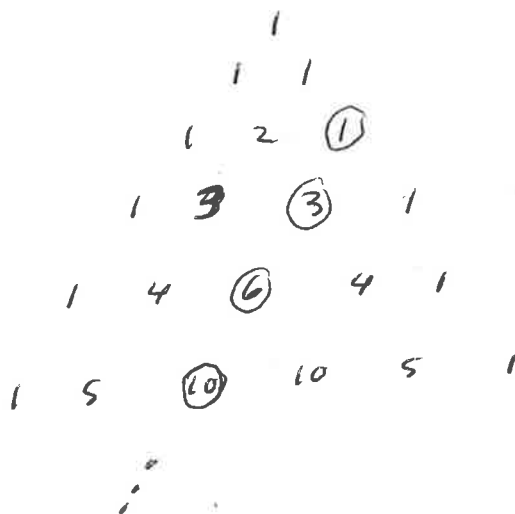
Pyramid of height n has $\binom{n+2}{3}$ balls □

In summary

$$\sum_{k=2}^N \binom{k}{2} = \binom{N+1}{3}$$

$$N = 2, 3, \dots$$

Interp using Pascals triangle



$$1 \quad n \quad \binom{n}{2} \quad \binom{n}{3} \quad \dots$$

$$1 \quad n+1 \quad \binom{n+1}{2} \quad \boxed{\binom{n+1}{3}} \quad \dots$$

the sum of the circled entries is boxed entry

Other diagonal sums are similarly handled:

For integers $0 \leq k \leq n$

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \dots + \binom{n}{k} = \binom{n+1}{k+1}$$

Ex For an integer $n \geq 1$ find

$$1^3 + 2^3 + 3^3 + \dots + n^3$$

*

in closed form

Sol Write n^3 in terms of binomial coeff. First observe

$$\binom{n}{3} = \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} = \frac{n^3 - 3n^2 + 2n}{6}$$

$$\binom{n}{2} = \frac{n(n-1)}{2} = \frac{n^2 - n}{2}$$

Solve for n^3 in terms of $\binom{n}{3}$, $\binom{n}{2}$, $\binom{n}{1} = n$

$$n^2 = 2 \binom{n}{2} + n$$

$$n^3 = 6 \binom{n}{3} + 3n^2 - 2n$$

$$= 6 \binom{n}{3} + 6 \binom{n}{2} + n$$

$$* = \sum_{k=1}^n k^3$$

$$= 6 \sum_{k=1}^n \binom{k}{3} + 6 \sum_{k=1}^n \binom{k}{2} + \sum_{k=1}^n k$$

$$= 6 \binom{n+1}{4} + 6 \binom{n+1}{3} + \binom{n+1}{2}$$

$$= \frac{(n+1)n(n-1)(n-2)}{4} + (n+1)n(n-1) + \frac{(n+1)n}{2}$$

$$= \frac{(n+1)n(n^2 - 3n + 2 + 4n - 2 + 2)}{4}$$

$$= \binom{n+1}{2}^2$$

□

For integers $l \geq 1$, $n \geq 1$

the sum

$$1^l + 2^l + 3^l + \dots + n^l$$

can be similarly found

We recall the binomial theorem:

thm Given variables x, y Given integer $n \geq 0$

$$\begin{aligned}(x+y)^n &= x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \dots + \binom{n}{n-1} x y^{n-1} + y^n \\ &= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k\end{aligned}$$

pf

Exp and LHS

$$(x+y)^n = (x+y)(x+y)(x+y) \dots (x+y) \quad n \text{ copies}$$

$$= \sum a_1 a_2 \dots a_n$$

sum is over all sequences $a_1, a_2, \dots, a_n$ such that

$$a_i \in \{x, y\} \text{ for } 1 \leq i \leq n$$

Each summand $a_1 a_2 \dots a_n$ contributes $x^{n-k} y^k$ where $k = |\{i \mid 1 \leq i \leq n, a_i = y\}|$

For a given k the number of summands $a_1 a_2 \dots a_n$

that contribute $x^{n-k} y^k$

$$= \# \text{ of ways to create } a_1 a_2 \dots a_n \text{ using exactly } k \text{ } y\text{'s}$$

$$= \# \text{ } k\text{-subsets of } \{1, 2, \dots, n\}$$

$$= \binom{n}{k}$$

Therefore

$$\sum a_1 a_2 \dots a_n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

□

Sp case: set $y=1$ in binom thm

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k}$$

$$= \sum_{k=0}^n \binom{n}{k} x^k$$

$$\left[\text{since } \binom{n}{k} = \binom{n}{n-k} \right]$$

Ex For an integer $n \geq 1$ find

$$\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \dots + n\binom{n}{n}$$

Sol

Consider

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

take the derivative with respect to x

$$n(1+x)^{n-1} = \sum_{k=0}^n \binom{n}{k} k x^{k-1}$$

$\downarrow$
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Now set $x=1$

$$n 2^{n-1} = \sum_{k=1}^n \binom{n}{k} k$$

$$= *$$



Ex For an integer $n \geq 1$ find

$$1^2 \binom{n}{1} + 2^2 \binom{n}{2} + 3^2 \binom{n}{3} + \dots + n^2 \binom{n}{n}$$

*

pf Consider

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Take deriv 2 times

$$n(1+x)^{n-1} \\ n(n-1)(1+x)^{n-2} = \sum_{k=0}^n \binom{n}{k} k(k-1) x^{k-2}$$

Set $x=1$

$$n(n-1)2^{n-2} = \sum_{k=0}^n \binom{n}{k} \underbrace{k(k-1)}_{k^2-k} \\ = \sum_{k=0}^n \binom{n}{k} k^2 - \underbrace{\sum_{k=0}^n \binom{n}{k} k}_{n2^{n-1}}$$

So

$$\sum_{k=0}^n \binom{n}{k} k^2 = n(n-1)2^{n-2} + n2^{n-1} \\ = n(n+1)2^{n-2}$$

For integers $k \geq 1$, $n \geq 1$ the sum

$$\sum_{k=0}^n k^k \binom{n}{k} \quad \text{can be similarly found}$$

□

