Math 846

Lecture 12

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter D.

thm 39 The intersection numbers c_i, a_i, b_i of Γ are determined by the spectrum

$$\begin{pmatrix} \theta_0, \theta_1, \dots \theta_D \\ m_0, m_1, \dots m_D \end{pmatrix}$$

as follows.

$$|X| = \sum_{i=0}^{D} m_i \qquad b_0 = k = \theta_0$$

polynomials

$$v_1 = \lambda \qquad u_1 = \lambda / k$$

$c_1 = 1$

$$a_1 = p^1_{11} = |X|^{-1} \sum_{\ell=0}^{D} v_1(\theta_\ell)\, v_1(\theta_\ell)\, u_1(\theta_\ell)\, m_\ell$$

$$b_1 = k - a_1 - 1$$

$$u_2 = \frac{\lambda u_1 - a_1 u_1 - c_1 u_0}{b_1}$$

$$c_2 = p^2_{11} = |X|^{-1} \sum_{\ell=0}^{D} v_1(\theta_\ell)\, v_1(\theta_\ell)\, u_2(\theta_\ell)\, m_\ell$$

$$k_2 = \frac{k b_1}{c_2} \qquad v_2 = u_2 k_2$$

$$a_2 = p^2_{21} = |X|^{-1} \sum_{\ell=0}^{D} v_2(\theta_\ell)\, v_1(\theta_\ell)\, u_2(\theta_\ell)\, m_\ell$$

$$b_2 = k - a_2 - c_2$$

$u_3 = \dots$ etc

$\square$

Recall the standard module $V = \mathbb{F}^X$

Recall for $0 \le j \le D$ the subspace $E_j V$ is the eigenspace of the adjacency matrix A with eigenvalue θ_j. We now examine the geometry of the vectors

$$\{ E_j \hat{x} \mid x \in X \}$$

Note that for $x, y \in X$ the following are the same:

(i) $\langle E_j \hat{x}, E_j \hat{y} \rangle$

(ii) $\langle \hat{x}, E_j \hat{y} \rangle$

(iii) coord x of $E_j \hat{y}$

(iv) $\langle E_j \hat{x}, \hat{y} \rangle$

(v) coord y of $E_j \hat{x}$

(vi) (x, y)-entry of E_j

LEM 40 Given DRG $\Gamma = (X, R)$ with diam D

For $0 \le i, j \le D$ and $x, y \in X$ at $\partial(x,y) = i$

(i) $\langle E_j \hat{x}, E_j \hat{y} \rangle = |X|^{-1} m_j u_i(\theta_j)$

(ii) $\| E_j \hat{x} \|^2 = |X|^{-1} m_j$

(iii) $$u_i(\theta_j) = \frac{\langle E_j \hat{x}, E_j \hat{y} \rangle}{\| E_j \hat{x} \| \, \| E_j \hat{y} \|}$$

$\qquad\qquad\qquad \|$

" the cosine of the angle between $E_j \hat{x}, E_j \hat{y}$ "

pf (i) Recall by Lem 20,

$$E_j = |X|^{-1} m_j \sum_{h=0}^{D} u_h(\theta_j) A_h$$

obs $\langle E_j \hat{x}, E_j \hat{y} \rangle = (E_j)_{xy} = |X|^{-1} m_j u_i(\theta_j)$

(ii) Set $i=0$ and $x=y$ in (i)

(iii) Combine (i), (ii) $\square$

Referring to Lem 40, we call the sequence $\{u_i(\theta_j)\}_{i=0}^{D}$ the <u>cosine sequence</u> for θ_j

Ex For a DRG $\Gamma = (X, R)$ diam D

Given scalars θ and $\{\sigma_i\}_{i=0}^{D}$ in $\mathbb{F}$

TFAE:

(i) θ is an eigenvalue of Γ with cosine sequence $\{\sigma_i\}_{i=0}^{D}$

(ii) $\sigma_0 = 1$ and

$$c_i \sigma_{i-1} + a_i \sigma_i + b_i \sigma_{i+1} = \theta \sigma_i \qquad (0 \le i \le D)$$

Another view:

LEM 41 With ref to Lem 40

$$\det \begin{pmatrix} \|E_j \hat{x}\|^2 & \langle E_j \hat{x}, E_j \hat{y} \rangle \\ \langle E_j \hat{y}, E_j \hat{x} \rangle & \|E_j \hat{y}\|^2 \end{pmatrix} = |X|^{-2} m_j^2 \left(1 - |u_i(\theta_j)|^2 \right)$$

$$= |X|^{-1} m_j \| E_j \hat{x} - u_i(\theta_j) E_j \hat{y} \|^2$$

$\square$

pf Use Lem 40.

COR 42 For a DRG $\Gamma = (X, R)$ with diam D,

$$|u_i(\theta_j)| \le 1 \qquad (0 \le i, j \le D)$$

pf By Lem 41 $\square$

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Cor 43 Given a DRG $\Gamma = (X, R)$ with diameter D. For $0 \le i, j \le D$ the following are equivalent:

(i) $u_i(\theta_j) = 1$

(ii) $E_j \hat{x} = E_j \hat{y}$ for all $x, y \in X$ at $\partial(x,y) = i$

(iii) $\exists$ $x, y \in X$ with $\partial(x,y) = i$ and $E_j \hat{x} = E_j \hat{y}$

pf Use Lem 41. $\square$

Cor 44 Given a DRG $\Gamma = (X, R)$ with diameter D. For $0 \le i, j \le D$ the following are equivalent:

(i) $u_i(\theta_j) = -1$

(ii) $E_j \hat{x} = - E_j \hat{y}$ for all $x, y \in X$ at $\partial(x, y) = i$

(iii) $\exists\, x, y \in X$ with $\partial(x, y) = i$ and $E_j \hat{x} = E_j \hat{y}$.

pf Use Lem 41. $\square$

Example 45 3-cube $H(3,2)$ $D=3$, $c_i = i$, $b_i = D-i$

Spectrum $\begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}$

i	0	1	2	3
u_i	1	$\frac{\lambda}{3}$	$\frac{\lambda^2-3}{6}$	$\frac{\lambda(\lambda^2-7)}{6}$

Table of cosines is

$$\begin{array}{c c} & \begin{array}{cccc} E_0 & E_1 & E_2 & E_3 \end{array} \\ \begin{array}{c} A_0 \\ A_1 \\ A_2 \\ A_3 \end{array} & \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{1}{3} & \frac{-1}{3} & -1 \\ 1 & \frac{-1}{3} & \frac{-1}{3} & 1 \\ 1 & -1 & 1 & -1 \end{pmatrix} \end{array}$$

(i,j)-entry is $u_i(\theta_j)$

Ex 45, cont. Take $\mathbb{F} = \mathbb{R}$

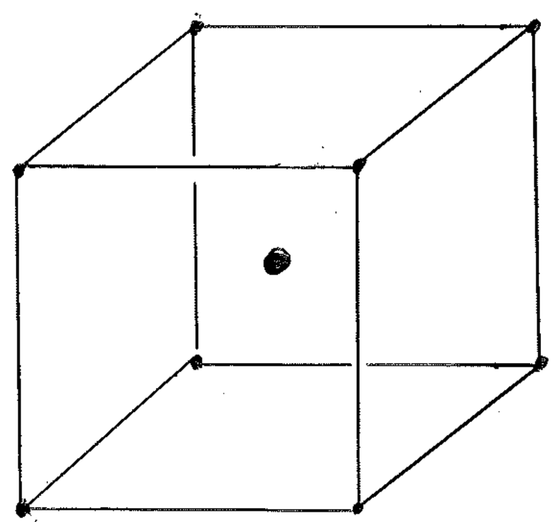
Consider cosines for $E = E_1$

$\dim EV = 3$

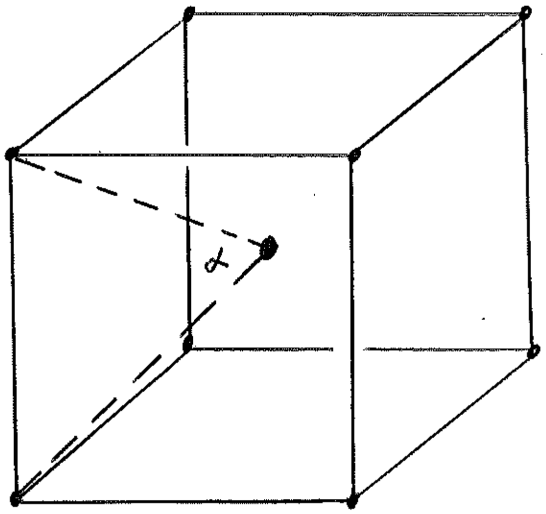
View EV as Euclidean 3-space

the vectors $\{ E\hat{x} \mid x \in X \}$ form the vertices

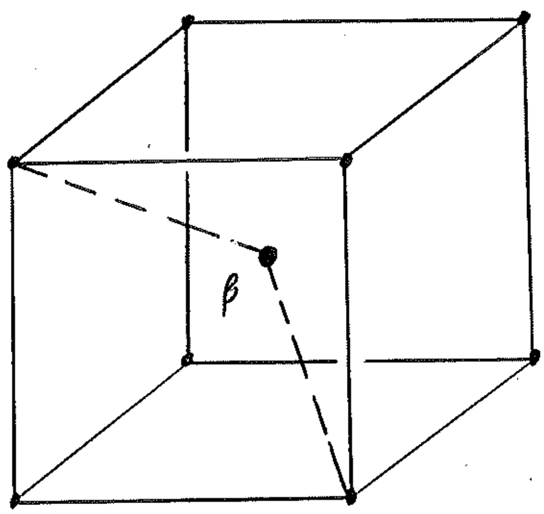
of a geometric cube centered at the origin



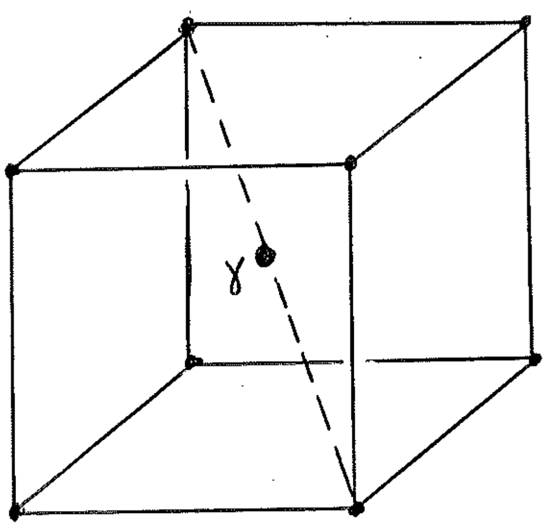
● = origin



$\cos \alpha = \frac{1}{3}$



$\cos \beta = -\frac{1}{3}$



$\cos \gamma = -1$

$\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ $\quad \Gamma = (X, \mathcal{R})$ is any DRG diam D.

DEF 46 A <u>representation</u> of Γ is a pair (ρ, H) where H is a nonzero Hermitean space (with inner product $\langle\, ,\, \rangle$) and $\rho : X \to H$ is a map such that

(R1) $\quad H = \operatorname{Span}\{ \rho(x) \mid x \in X \}$

(R2) $\quad \forall\, x, y \in X,$

$\langle \rho(x), \rho(y) \rangle$ depends only on $\partial(x, y)$

(R3) $\quad \forall x \in X,$

$$\sum_{y \in \Gamma(x)} \rho(y) \in \operatorname{Span}(\rho(x))$$

The above representation is <u>nondegenerate</u> whenever $\{ \rho(x) \mid x \in X \}$ are mutually distinct

Ex 47 For the D-cube $\Gamma = H(D,2)$ view

$$X = \left\{ (\alpha_1, \alpha_2, \dots, \alpha_D) \mid \alpha_i \in \{1, -1\} \quad 1 \le i \le D \right\}$$

Take $H = \mathbb{F}^D$ (column vectors) $\qquad \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$\langle u, v \rangle = u^t \bar{v} \qquad (u, v \in H)$$

For a vertex $x = (\alpha_1, \alpha_2, \dots, \alpha_D)$ of Γ define

$$\rho(x) = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_D \end{pmatrix} \in H$$

Then (ρ, H) is a representation of Γ