

Combinatorics 475

Lecture 11

4.5 Partial Orders and Equivalence Relations

Consider a set X

Consider the Cartesian product

$$X \times X = \{(a, b) \mid a, b \in X\}$$

A subset

$$R \subseteq X \times X$$

is called a relation on X

For $a, b \in X$ write

$$a R b \quad \text{whenever} \quad (a, b) \in R$$

"a is related to b"

$$a \not R b \quad \dots \quad (a, b) \notin R$$

"a not related to b"

Ex 1 $X =$ set of integers

$$\forall a, b \in X$$

$$a R b \quad \text{whenever} \quad a - b \text{ is even}$$

Ex 2 $X =$ set of all subsets of $\{1, 2, \dots, n\}$

$$\forall a, b \in X$$

$$a R b \quad \text{whenever} \quad a \subseteq b$$

$$Ex 3 \quad X = \{1, 2, \dots, 100\}$$

$$\forall a, b \in X$$

$$a R b \text{ whenever } a < b$$

We now list some important conditions on relations.
Given a relation R on a set X ,

Condition	Meaning	examples
Reflexive	$x R x$ (for all $x \in X$)	1, 2
Irreflexive	$x \not R x$ (for all $x \in X$)	3
Symmetric	$x R y$ implies $y R x$ (for all $x, y \in X$)	1
Antisymmetric	For distinct $x, y \in X$, $x R y$ implies $y \not R x$	2, 3
Transitive	$x R y$ and $y R z$ implies $x R z$ (for all $x, y, z \in X$)	1, 2, 3

Def Given a relation R on a set X .

- Call R a partial order whenever R is reflexive, antisymmetric, transitive.
- Call R a strict partial order whenever R is Irreflexive, antisymmetric, transitive.
- The set X together with a partial order on X is called a partially ordered set (poset)
- For $x, y \in X$, call x, y comparable whenever xRy or yRx
Call x, y incomparable whenever $x \not R y$ and $y \not R x$
- Call R a total order whenever R is a partial order and x, y are comparable for all $x, y \in X$

Ex Given integer $n \geq 1$

$$X = \{1, 2, \dots, n\}$$

Given permutation $a_1, a_2, \dots, a_n$ of X

Define a relation R on X by:

$$\forall x, y \in X$$

$x R y$ whenever $x = y$ or x comes before y among $a_1, a_2, \dots, a_n$

Then R is a total order (check)

Ex Given integer $n \geq 1$

$$X = \{1, 2, \dots, n\}$$

Given total order R on X

Call an element $x \in X$ minimal whenever there does not exist $y \in X$ s.t. $y \neq x$ and $y R x$

Observe

X has unique minimal element a_1

$$X \setminus \{a_1\} \quad \dots \quad a_2$$

$$X \setminus \{a_1, a_2\} \quad \dots \quad a_3$$

$\vdots$

Then $a_1, a_2, \dots, a_n$ is permutation of X .

The previous 2 examples show:

Then Given integer $n \geq 1$

$$X = \{1, 2, \dots, n\}.$$

There is a 1-1 correspondence between the total orders on X and the permutations of X .

— 0 —

A generic partial order on a set X is usually denoted $\leq$.

For $x, y \in X$ write $x < y$ whenever $x \neq y$ and $x \leq y$.

Def Given a set X with partial order $\leq$

For $x, y \in X$ we say y covers x whenever

$x < y$ and there does not exist $z \in X$ such that

$$x < z < y$$

In this case write

$$x <_c y$$

Ex Take $X = \{1, 2, 3, \dots, 10\}$

Define a partial order $\leq$ on X by

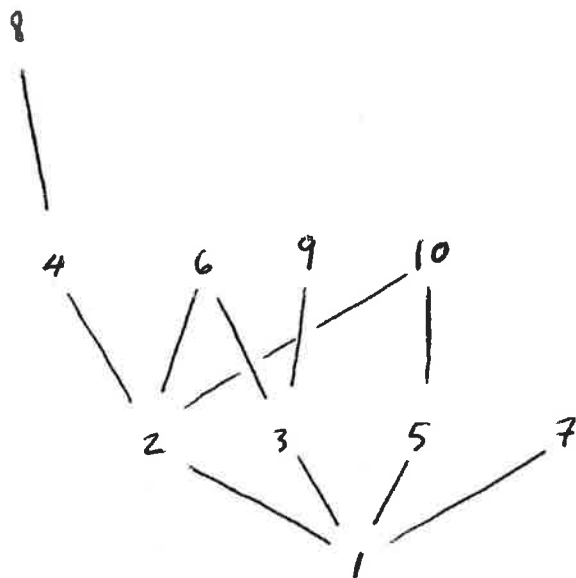
$x \leq y$ whenever $x \mid y$ (x divides y)

Describe the cover relation $<_c$

Sol. For $x, y \in X$ draw

$$\begin{array}{c} y \\ | \\ x \end{array}$$

whenever $x <_c y$



"Hasse diagram"

Given a set S

let $\mathcal{P}(S)$ denote the set of all subsets of S

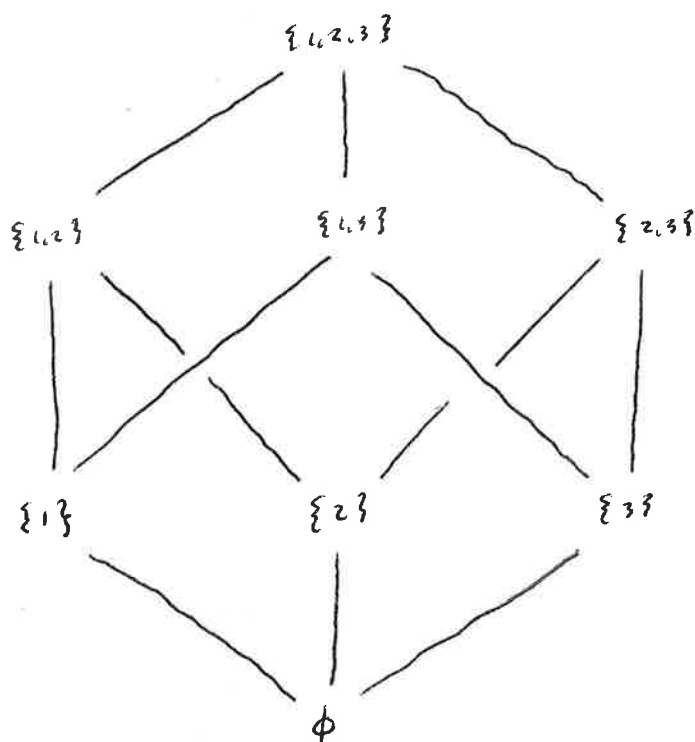
$$|\mathcal{P}(S)| = 2^n \quad n = |S|$$

Obs the containment relation $\subseteq$ is a partial order on $\mathcal{P}(S)$

Ex Take $S = \{1, 2, 3\}$

$$X = \mathcal{P}(S)$$

Describe the Hasse diagram for the poset $X, \subseteq$



It's the 3-cube!

Def Given set X

Given 2 partial orders on X :

$$\leq_1 \qquad \leq_2$$

$\leq_2$ is called an extension of $\leq_1$ whenever

$$x \leq_1 y \text{ implies } x \leq_2 y \qquad \forall x, y \in X$$

Def Given a partial order $\leq$ on a set X

A linear extension of $\leq$ is an extension of $\leq$

that is a total order

Thm Let $\leq$ denote a partial order on a finite set X .
then $\leq$ has at least one linear extension.

p.f. Let $n = |X|$

Let x_1 denote a minimal element of X

$$X \setminus \{x_1\}$$

Let $x_2 \quad \dots$

$$X \setminus \{x_1, x_2\}$$

Let $x_3 \quad \dots$

$\vdots$

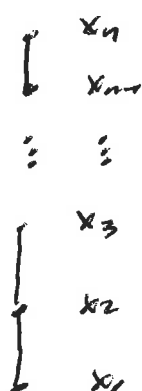
This gives ordering $x_1, x_2, \dots, x_n$ of elements of X

Define a total order on X as follows.

In the total order

$$x_i \text{ covers } x_{i+1} \text{ for } 1 \leq i \leq n$$

So Hasse diagram is

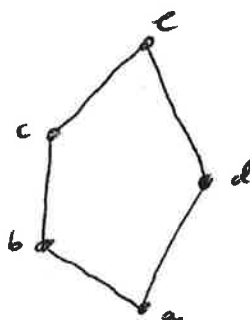


One checks the total order is a linear ext of $\leq$

□

Ex

Consider poset with Hasse diagram



Find all the linear extensions.

Sol

$$\begin{aligned} a &< b < c < d < e \\ a &< b < d < c < e \\ a &< d < b < c < e \end{aligned}$$

□

We now discuss equivalence relations

Given set X

A relation R on X is an equivalence relation whenever R is

reflexive, symmetric, transitive

Ex $S = \{1, 2, \dots, n\}$

$$X = \mathcal{P}(S)$$

Define a relation R on X by

$$x R y \text{ whenever } |x| = |y|$$

then R is an equiv relation

— o —

Ex Given a set X

Partition X into nonempty subsets

$$X = X_1 \cup X_2 \cup \dots \cup X_r \quad (\text{disjoint union})$$

Define a relation R on X as follows:

$\forall x, y \in X$

$x R y$ whenever x, y are in the same part of the partition

Then R is an equiv relation (check)

Ex Given an equiv relation R on a set X

$\forall x \in X$ define

$$[x] = \{ y \in X \mid x R y \}$$

"the equivalence class of x "

Obs $\forall x, y \in X$

$$[x] = [y] \quad \text{if } x R y$$

$$\text{or } [x] \cap [y] = \emptyset \quad \text{if } x \not R y$$

Moreover the set of distinct equiv classes give a partition of X into nonempty subsets.

A generic equivalence relation is usually denoted by $\sim$

