Math 846

Lecture 11

LEM 26 For a DRG $\Gamma = (X, R)$ with diameter D,

(i) $u_i(\theta_0) = 1 \qquad (0 \le i \le D)$

(ii) $v_i(\theta_0) = k_i \qquad (0 \le i \le D)$

pf (i) By Lem 20,

$$E_0 = |X|^{-1} \underset{\substack{\shortparallel \\ 1}}{m_0} \sum_{i=0}^{D} u_i(\theta_0) A_i$$

By Lem 25,

$$E_0 = |X|^{-1} J$$
$$= |X|^{-1} \sum_{i=0}^{D} A_i$$

Result follows since $\{A_i\}_{i=0}^{D}$ are lin indep.

(ii) By Lem 18

$v_i = k_i u_i$

$\square$

More intersection numbers.

Given a DRG $\Gamma = (X, R)$ with diameter D, since the distance matrices $\{A_i\}_{i=0}^{D}$ form a basis for M, $\exists$ scalars

$$p^h_{ij} \in \mathbb{F} \qquad (0 \le h, i, j \le D)$$

such that

$$A_i A_j = \sum_{h=0}^{D} p^h_{ij} A_h \qquad (*)$$

for $0 \le i, j \le D$. Since A_i, A_j commute,

$$p^h_{ij} = p^h_{ji} \qquad (0 \le h, i, j \le D)$$

For $0 \le h \le D$ and $x, y \in X$ at $\partial(x,y) = h$, from the (x,y)-entries in $(*)$ we obtain

$$p^h_{ij} = \left| \Gamma_i(x) \cap \Gamma_j(y) \right|$$

So $p^h_{ij} \in \mathbb{N}$. For notational convenience define

$$p^h_{ij} = 0 \quad \text{unless} \quad 0 \le i, j \le D \qquad (i, j \in \mathbb{Z})$$

Note that

$$c_i = p^i_{1,i-1} \qquad a_i = p^i_{1i} \qquad b_i = p^i_{1,i+1}$$

for $0 \le i \le D$. We often call any p^h_{ij} an <u>intersection number</u> of Γ. By simple counting arguments,

$$p^h_{0j} = \delta_{hj} \qquad (0 \le h, j \le D)$$

$$p^h_{i0} = \delta_{hi} \qquad (0 \le h, i \le D)$$

$$p^0_{ij} = \delta_{ij} k_i \qquad (0 \le i, j \le D)$$

$$\sum_{i=0}^{D} p^h_{ij} = k_j \qquad (0 \le h, j \le D)$$

LEM 27 With above notation,

$$k_h p^h_{ij} = k_i p^i_{jh} = k_j p^j_{hi} \qquad (0 \le h, i, j \le D)$$

pf Each of the three products is $|X|^{-1}$ times the number of triples

$$\left| \left\{ xyz \mid x, y, z \in X,\ \partial(x,y) = h,\ \partial(y,z) = i,\ \partial(x,z) = j \right\} \right|$$

$\square$

LEM 28 For a DRG Γ with diameter D and for $0 \le h, i, j \le D$

(i) $p^h_{ij} = 0$ if one of h, i, j is greater than the sum of the other two

(ii) $p^h_{ij} \ne 0$ if one of h, i, j is equal to the sum of the other two

pf The distance function ∂ satisfies the triangle inequality. $\square$

Next goal: Show each p^h_{ij} is determined by $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$

LEM 29 For a DRG Γ with diameter D

and for $0 \le h, i, j, l \le D$,

$$\sum_{\alpha=0}^{D} p^{l}_{h\alpha}\, p^{\alpha}_{ij} = \sum_{\beta=0}^{D} p^{l}_{j\beta}\, p^{\beta}_{hi}$$

pf Expand each side of

$$A_h (A_i A_j) = (A_h A_i) A_j$$

as a linear combination of distance matrices, and compare coefficients. $\square$

LEM 30 For a DRG Γ with diameter D

and for $0 \le l, i, j \le D$

$$c_l p^{l-1}_{ij} + a_l p^{l}_{ij} + b_l p^{l+1}_{ij} = b_{i-1} p^{l}_{i-1,j} + a_i p^{l}_{ij} + c_{i+1} p^{l}_{i+1,j}$$

$\square$

pf Take $h=1$ in LEM 29.

<u>Note</u> For a DRG Γ with diameter D we can compute

$$p^h_{ij} \qquad 0 \le h, i, j \le D$$

from $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ using the recursion in LEM 30

Ex 31 For a DRG Γ with diameter D,

$$p^i_{2,i-1} = c_i \frac{a_i + a_{i-1} - a_1}{c_2} \qquad 1 \le i \le D$$

$$p^i_{2,i+1} = b_i \frac{a_i + a_{i+1} - a_1}{c_2} \qquad 0 \le i \le D-1$$

$$p^i_{2i} = \frac{c_i(b_{i-1}-1) + a_i(a_i - a_1 - 1) + b_i(c_{i+1}-1)}{c_2} \qquad 0 \le i \le D$$

For a DRG $\Gamma = (X, R)$ with diameter D,

recall the polynomials $\{u_i\}_{i=0}^{D}$, $\{v_i\}_{i=0}^{D}$ from

Def 17, Def 10.

We wish to say that these are <u>orthogonal polynomials.</u>

What does this mean? What is orthogonal to what?

let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

DEF 32 We endow the vector space $Mat_X(\mathbb{F})$

with a function

$$\langle B, C \rangle = \operatorname{trace}(B \bar{C}^t) \qquad B, C \in Mat_X(\mathbb{F})$$

obs the pair $Mat_X(\mathbb{F})$, $\langle\, , \rangle$ is a

Hermitean space according to Def 23 in Chapter 1.

LEM 33 Referring to Def 32.

$$\langle AB, C \rangle = \langle B, \bar{A}^t C \rangle$$
$$= \langle A, C \bar{B}^t \rangle$$

for all $A, B, C \in Mat_{\times}(\mathbb{F})$.

pf Use $tr(RS) = tr(SR)$ $\square$

Lem 34 For a DRG with diameter D,

(i) $\langle A_i, A_j \rangle = \delta_{ij} k_i |X| \qquad (0 \le i,j \le D)$

(ii) $\langle E_i, E_j \rangle = \delta_{ij} m_i \qquad (0 \le i,j \le D)$

pf (i)

$$\begin{aligned}
\langle A_i, A_j \rangle &= \langle A_i, I A_j \rangle \\
&= \langle A_i \bar{A}_j^t, I \rangle \\
&= \langle A_i A_j, I \rangle \\
&= \sum_{h=0}^{D} p^h_{ij} \underbrace{\langle A_h, I \rangle}_{\mathrm{tr}(A) \,=\, \delta_{h,0} |X|} \\
&= p^0_{ij} |X| \\
&= \delta_{ij} k_i |X|
\end{aligned}$$

(ii)

$$\begin{aligned}
\langle E_i, E_j \rangle &= \langle E_i, I E_j \rangle \\
&= \langle E_i \bar{E}_j^t, I \rangle \\
&= \langle E_i E_j, I \rangle \\
&= \delta_{ij} \langle E_i, I \rangle \\
&= \delta_{ij} \mathrm{tr}(E_i) \\
&= \delta_{ij} m_i
\end{aligned}$$

□

thm 35 For a DRG $\Gamma = (X, \mathcal{R})$ with diameter D,

(i) For $0 \le i, j \le D$

$$\sum_{\ell=0}^{D} v_i(\theta_\ell)\, v_j(\theta_\ell)\, m_\ell = \delta_{ij}\, k_i\, |X|$$

"row orthogonality"

(ii) For $0 \le s, t \le D$,

$$\sum_{i=0}^{D} v_i(\theta_s)\, v_i(\theta_t)\, k_i^{-1} = \delta_{st}\, m_s^{-1}\, |X|$$

"column orthogonality"

pf (i) In the equation

$$\langle A_i, A_j \rangle = \delta_{ij}\, k_i\, |X|$$

write each of A_i, A_j as a linear combination of the primitive idempotents using Lem 20, and simplify using Lem 34 (ii).

(ii) In the equation

$$\langle E_s, E_t \rangle = \delta_{st}\, m_s$$

write each of E_s, E_t as a linear combination of distance matrices using Lem 20, and simplify using Lem 34 (i)

$\square$

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Thm 36 For a DRG $\Gamma = (X, R)$ with diameter D,

(i) For $0 \le i, j \le D$

$$\sum_{l=0}^{D} u_i(\theta_l)\, u_j(\theta_l)\, m_l = \delta_{ij}\, k_i^{-1} |X|$$

(ii) For $0 \le s, t \le D$,

$$\sum_{i=0}^{D} u_i(\theta_s)\, u_i(\theta_t)\, k_i = \delta_{st}\, m_s^{-1} |X|$$

pf Evaluate Thm 35 using $v_i = u_i k_i \quad (0 \le i \le D)$. $\square$

Another formula for p^h_{ij}

LEM 37. For a DRG $\Gamma = (X, \mathcal{R})$ with diameter D

and for $0 \le h, i, j \le D$,

$$p^h_{ij} = |X|^{-1} k_h^{-1} \langle A_i A_j, A_h \rangle$$

$$= |X|^{-1} k_h^{-1} \langle A_h, A_i A_j \rangle$$

pf Expand $A_i A_j$ using

$$A_i A_j = \sum_{\ell=0}^{D} p^\ell_{ij} A_\ell$$

and use Lem 34 (i). $\square$

Thm 38 For a DRG $\Gamma = (X, R)$ with diameter D

and for $0 \le h, i, j \le D$,

$$p^h_{ij} = |X|^{-1} \sum_{\ell=0}^{D} v_i(\theta_\ell) v_j(\theta_\ell) u_h(\theta_\ell) m_\ell$$

pf Use Lem 37 and

$$\langle A_i A_j, A_h \rangle = \left\langle \sum_{\ell=0}^{D} v_i(\theta_\ell) v_j(\theta_\ell) E_\ell, \sum_{t=0}^{D} v_h(\theta_t) E_t \right\rangle$$

$$= \sum_{\ell=0}^{D} v_i(\theta_\ell) v_j(\theta_\ell) v_h(\theta_\ell) m_\ell$$

$\square$