

Combinatorics 475

Lecture 10

We just showed how to List the subsets of an n -element set.

We now give an alt approach using Gray codes

Problem For $n \geq 1$ list all the n -tuples of zeros and ones such that

each n -tuple differs from previous one in exactly one coordinate

such a List is called a Gray code of order n

ex $n = 3$

```

000
|
001
|
011
|
010
|
110
|
111
|
101
|
100

```

Def. A Gray code is cyclic whenever 1st term and Last term differ in exactly one coord

Geometric interpretation of Gray codes

For an integer $n \geq 1$ we define a graph called the n -cube

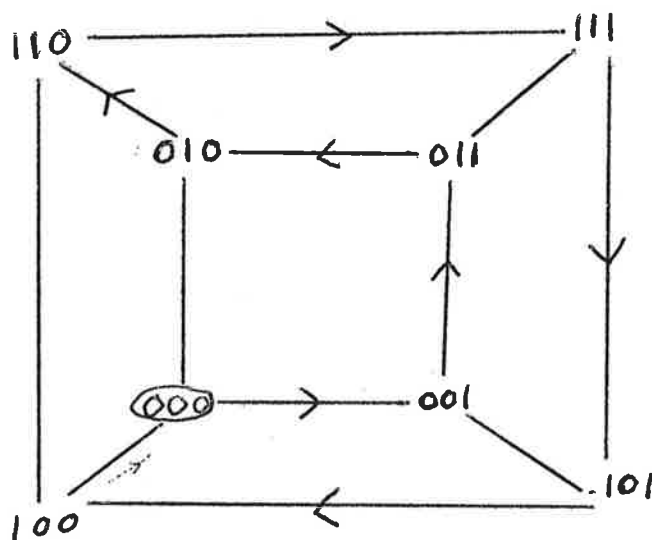
- The vertex set X consists of all n -tuples of 0's and 1's.
so $|X| = 2^n$
- Two vertices declared adjacent whenever they differ in exactly one coordinate

n	n -cube
1	0 — 1
2	<pre> 10 — 11 00 — 01 </pre>
3	<pre> 110 — 111 / \ 010 — 011 000 — 001 / \ 100 — 101 </pre>
⋮	⋮

Observe

- A Gray Code of order n is a path through the n -cube that visits each vertex exactly once.
- the Gray Code is cyclic whenever the last vertex is adjacent to the first vertex.

ex $n=3$



○ = start

cyclic
Gray Code

We now describe a special
cyclic Gray code called the

Reflected Gray Code

ex display the reflected Gray code of order 4

coords				
3	2	1	0	
0	0	0	0	
0	0	0	1	
0	0	1	1	reflect
0	0	1	0	
0	1	1	0	reflect
0	1	1	1	
0	1	0	1	
0	1	0	0	reflect
1	1	0	0	
1	1	0	1	
1	1	1	1	
1	1	1	0	
1	0	1	0	
1	0	1	1	
1	0	0	1	
1	0	0	0	

Algorithm to obtain the reflected Gray code of order n :

- Reflected Gray code of order $n=1$ is

0
1
- For $n \geq 2$ the reflected Gray code of order n is obtained from the reflected Gray code of order $n-1$ by creating 2 copies, with 2nd copy in inverted order.

For each term in Copy 1	add a leading 0
For each term in Copy 2	add a leading 1

— 0 —

4.4 Generating r -subsets

Problem Given a finite set S .

Given integer r $1 \leq r \leq |S|$

List all the r -subsets of S .

Sol 1: List all the subsets of S using the method of Section 4.2.

Discard each term that does not have cardinality r .

Sol 2: Use Lexicographical order as described below.

Ex $|S| = 26$ $r = 3$

View S as the letters of the alphabet

$$S = \{a, b, c, \dots, z\}$$

View each 3-subset of S as a word of length 3 and list them in alphabetical order:

abc

abd

abe

:

abz

acd

ace

:

acz

ade

:

:

ayz

bcd

:

byz

cde

:

xyz

Lex order

Ex In Lex order, list all the
3-words of $\{1, 2, 3, 4, 5\}$

Sol Consider alphabet with "letters"

$$1 < 2 < 3 < 4 < 5$$

In alphabetical order List all words of form
xyz

$$1 \leq x < y < z \leq 5$$

123

124

125

134

135

145

234

235

245

345

↓ Lex order

□

Ex Consider the 5-subsets of

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

in Lex order.

What is position of

134~~7~~8

(*)

ie if (*) is the m th term, what is m ?

[start counting with $m=1$]

Sol

strategy: count the terms that come after (*).

types of 5-subsets			# choices
No restr:	abcde	$1 \leq a < b < c < d < e \leq 9$	$\binom{9}{5}$
come after (*)	abcde	$2 \leq a < b < c < d < e \leq 9$	$\binom{8}{5}$
	1bcde	$4 \leq b < c < d < e \leq 9$	$\binom{6}{4}$
	13cde	$5 \leq c < d < e \leq 9$	$\binom{5}{3}$
	134de	$8 \leq d < e \leq 9$	$\binom{2}{2} = 1$
	134 ~~7~~ e	$9 \leq e \leq 9$	$\binom{1}{1} = 1$

$$m = \binom{9}{5} - \binom{8}{5} - \binom{6}{4} - \binom{5}{3} - \binom{2}{2} - \binom{1}{1}$$

thm For $1 \leq r \leq n$ consider r -subsets of
 $\{1, 2, \dots, n\}$

in lex order.

The pos of the r -subset

$$\{a_1, a_2, \dots, a_r\}$$

$$1 \leq a_1 < a_2 < \dots < a_r \leq n$$

is

$$\binom{n}{r} - \binom{n-a_1}{r} - \binom{n-a_2}{r-1} - \dots - \binom{n-a_{r-1}}{2} - \binom{n-a_r}{1}$$

pf Same ideas as in prec example

□

Problem Given a finite set S

Given integer r $1 \leq r \leq |S|$

List all the r -permutations of S

Sol: First List the r -subsets of S in Lex order
For each subset list all the permutations of its
elements using the method of Section 4.1

ex List all the 3-perms of $\{1, 2, 3, 4\}$.

Sol First List 3-subsets:

123

124

134

234

Expand:

123

132

312

321

231

213

124

142

412

421

241

214

134

143

413

431

341

314

234

243

423

432

342

324

□

We continue to discuss the reflected Gray codes

Problem Given a term in the reflected Gray code of order n :

$$g_{n-1} \dots g_2 g_1 g_0$$

- (i) what is next term?
- (ii) what is preceding term?

If we list the code words for small values of n the following pattern emerges:

For (i), (ii) we just need to specify which word to change:

(i) For $\sum_{i=0}^{n-1} g_i$ even change g_0

For $\sum_{i=0}^{n-1} g_i$ odd change g_1 for the unique

integer $s \geq 1$ such that $g_{s-1} = 1$ and each of $g_0, g_1, \dots, g_{s-2}$ is 0

$$\dots ** \boxed{1} 00 \dots 0$$

↑
change

(ii) For $\sum_{i=0}^{n-1} g_i$ odd change g_0

For $\sum_{i=0}^{n-1} g_i$ even change g_s for the unique

integer $s \geq 1$ such that $g_{s-1} = 1$ and each of

$g_0, g_1, \dots, g_{s-2} = 0$

Ex Consider reflected Gray code of order 8

For the term

i	7	6	5	4	3	2	1	0
g_i	1	0	1	0	0	1	1	0

(i) Find next term

(ii) Find preceding term

Note $\sum g_i = 4$ is even

(i) change g_0 :

ans = 10100111

(ii) Find s such that
 $g_s g_{s-1} \dots g_0 g_0 =$

$\square 10 \dots 00$

$s=2$:

$\boxed{1} 1 0$

↑

change

ans =

10100010

$\square$

