Math 846

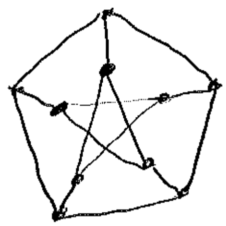
Lecture 10

We continue to discuss a DRG $\Gamma = (X, R)$ with diameter D. We seek to find the spectrum

$$\begin{pmatrix} \theta_0, \theta_1, \ldots, \theta_D \\ m_0, m_1, \ldots, m_D \end{pmatrix}$$

in terms of the intersection numbers of Γ. So far we found $\{\theta_i\}_{i=0}^{D}$ as the roots of a polynomial $v_{D+1}(\lambda)$.

Ex 12 Petersen graph



is DRG with $D=2$ and intersection numbers

$c_1 = 1$ $c_2 = 1$

$a_0 = 0$ $a_1 = 0$ $a_2 = 2$

$k = 3$ $b_1 = 2$

Here

$v_0 = 1$ $v_1 = \lambda$ $v_2 = \lambda^2 - 3$

$$\begin{aligned} v_3 &= \lambda^3 - 2\lambda^2 - 5\lambda + 6 \\ &= (\lambda - 3)(\lambda - 1)(\lambda + 2) \end{aligned}$$

Distinct eigvals of Γ are

$\theta_0 = 3,$ $\theta_1 = 1,$ $\theta_2 = -2$ $\square$

A DRG with diameter $D=2$ is called <u>strongly regular</u> (SRG).

For a DRG we now find the multiplicities

m_i in terms of the intersection numbers.

DEF 13 For a DRG $\Gamma = (X, \mathcal{R})$ with diameter D

define

$$B = \begin{pmatrix} a_0 & b_0 & & & & 0 \\ c_1 & a_1 & b_1 & & & \\ & c_2 & a_2 & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & b_{D-1} \\ 0 & & & & c_D & a_D \end{pmatrix} \qquad (D+1) \times (D+1)$$

Call B the <u>intersection matrix</u> of Γ

LEM 14 With above notation, the following coincide

(i) the min polynomial of A

(ii) the min polynomial of B

(iii) the char polynomial of B

pf (i) = (ii) B represents the action of A on $\{A_i\}_{i=0}^{D}$ and this action is faithful.

(ii) = (iii) min poly of B has degree $D+1$ □.

COR 15 With above notation,
the eigenvalues of B are mutually distinct and are precisely the dist eigenvalues of A. □

LEM 16 With above notation, for an eigenvalue θ of B define the row vector

$$v = \left(v_0(\theta), v_1(\theta), \dots, v_D(\theta) \right)$$

where v_i is from DEF 10. Then

$$vB = \theta v$$

pf Use DEF 10 and DEF 13. $\square$

DEF 17 Given a DRG $\Gamma = (X, R)$ with diameter D define polynomials $\{u_i\}_{i=0}^{D}$ in $\mathbb{F}[\lambda]$ by

$$u_0 = 1, \qquad u_1 = \frac{\lambda}{k}$$

$$\lambda u_i = c_i u_{i-1} + a_i u_i + b_i u_{i+1} \qquad (1 \le i \le D-1)$$

LEM 18 With above notation,

$$u_i = \frac{v_i}{k_i} \qquad (0 \le i \le D)$$

pf Define $v_i' = k_i u_i \qquad 0 \le i \le D.$

Obs $\quad v_0' = 1, \qquad v_1' = \lambda = v_1$

Using LEM 4 and DEF 17 we find

$$\lambda v_i' = c_{i+1} v_{i+1}' + a_i v_i' + b_{i-1} v_{i-1}' \qquad 1 \le i \le D-1.$$

Compare this to DEF 10 and find $v_i' = v_i \quad (0 \le i \le D)$. $\square$

LEM 19 With above notation. for an eigenvalue θ of B define the column vector

$$u = \begin{pmatrix} u_0(\theta) \\ u_1(\theta) \\ \vdots \\ u_D(\theta) \end{pmatrix}$$

Then

$$Bu = \theta u$$

pf Use DEF 13 and DEF 17 $\square$

LEM 20 With above notation.

(i) $$A_j = \sum_{i=0}^{D} v_j(\theta_i) E_i \qquad (0 \le j \le D)$$

(ii) $$E_j = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \qquad (0 \le j \le D)$$

pf (i) Recall

$$A = \sum_{i=0}^{D} \theta_i E_i$$

so

$$\underset{\substack{\| \\ A_j}}{v_j(A)} = \sum_{i=0}^{D} v_j(\theta_i) E_i$$

(ii) Write $S =$ sum in RHS

show $E_j = S$

First show

$$A S = \theta_j S$$

$$A S = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) \underbrace{A A_i}_{= c_{i+1} A_{i+1} + a_i A_i + b_{i-1} A_{i-1}}$$

$$= |X|^{-1} m_j \sum_{l=0}^{D} A_l \Big(\underbrace{c_l u_{l-1}(\theta_j) + a_l u_l(\theta_j) + b_l u_{l+1}(\theta_j)}_{= \theta_j u_l(\theta_j)} \Big)$$

$$= \theta_j S$$

We have $A S = \theta_j S$ and $S \in M$

so $S = \alpha E_j$ for some $\alpha \in \mathbb{F}$

show $\alpha = 1$

Take trace

$$\mathrm{tr}(S) = \alpha \underbrace{\mathrm{tr}(E_j)}_{= m_j}$$

obs $\mathrm{tr}(A_l) = \delta_{l0} |X| \qquad 0 \le l \le D$

so $\mathrm{tr}(S) = m_j$ and $\alpha = 1$

So $E_j = S$. $\square$

thm 21 (Norman Biggs 1974)

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For a DRG $\Gamma = (X, R)$ with diameter D,

for an eigenvalue θ of Γ, the corresponding

multiplicity is

$$\frac{|X|}{\sum_{i=0}^{D} u_i(\theta)\, v_i(\theta)}$$

pf Write $\theta = \theta_j$. Observe

$$\begin{aligned} E_j &= E_j^2 \\ &= \left(|X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j) A_i \right) E_j \\ &= |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(A)\, E_j \\ &= |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(\theta_j)\, E_j \end{aligned}$$

So

$$1 = |X|^{-1} m_j \sum_{i=0}^{D} u_i(\theta_j)\, v_i(\theta_j).$$

Result follows.

$\square$

COR 22 The spectrum of a DRG is determined by its intersection numbers.

pf let

$$\begin{pmatrix} \theta_0 & \theta_1 & \cdots & \theta_D \\ m_0 & m_1 & \cdots & m_D \end{pmatrix}$$

be the spectrum in question.

Given the intersection numbers c_i, a_i, b_i

we get $\{v_i\}_{i=0}^{D+1}$ by Def 10

we get $\{\theta_i\}_{i=0}^{D}$ as the roots of v_{D+1}

we get $\{u_i\}_{i=0}^{D}$ by Def 17

we get $\{k_i\}_{i=0}^{D}$ by $k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$

we get $|X|$ by $|X| = \sum_{i=0}^{D} k_i$

we get $\{m_i\}_{i=0}^{D}$ by thm 21.

$\square$

Ex 23 Peterson's graph revisited

Using Cor 22 the spectrum is

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}$$

Note Given pos integers $\{c_i\}_{i=1}^{D}$, $\{b_i\}_{i=0}^{D-1}$ does there exist a corresponding DRG Γ?

Compute the spectrum of Γ using Cor 22.

If the m_i fail to be positive integers then Γ does not exist.

This is a "feasibility condition" on the intersection numbers.

Ex 24 Show that there does not exist a DRG with diameter $D = 3$ and intersection numbers

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$$c_1 = 1 \qquad c_2 = 1 \qquad c_3 = 3$$

$$k = 3 \qquad b_1 = 2 \qquad b_2 = 1$$

$\square$

pf ex

We have a few comments.

LEM 25 For a DRG $\Gamma = (X, R)$

(i) $E_0 = |X|^{-1} J$ $\quad$ (J = all 1's)

(ii) $E_0 V = \mathrm{Span}(\mathbb{1})$ $\quad$ ($\mathbb{1} = \sum_{x \in X} \hat{x}$)

pf (i) Write $E = |X|^{-1} J$

obs $E = |X|^{-1} \sum_{i=0}^{D} A_i \in M$

Also $E^2 = E$ since $J^2 = |X| J$

Also $AE = \theta_0 E$ $\quad$ $\theta_0 = k$ = valency of Γ

So $E = E_0$

(ii) Clear

$\square$