LECTURE 9 MONDAY SEPT. 23

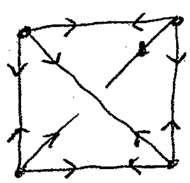
9/23/13

In lecture 8 we turned the standard module for the hypercube Q_D into a $\boxtimes$-module.

Shortly we will consider more general $\boxtimes$-modules.

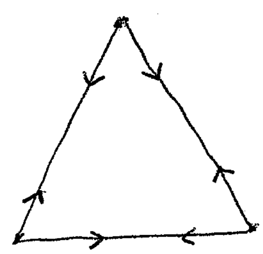
First we consider sl_2

Recall our presentation of $\boxtimes$:



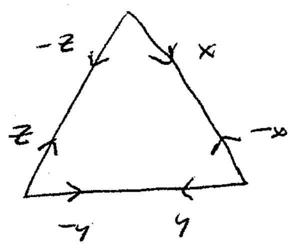
each $\boxtimes$ generator x_{rs} is represented by the directed arc from node r to node s

Consider a "face" of the tetrahedron



this is sl_2 in disguise

Label the generators



$$[x,y] = 2x+2y \qquad (1)$$

$$[y,z] = 2y+2z \qquad (2)$$

$$[z,x] = 2z+2x \qquad (3)$$

Let L denote the Lie algebra over $\mathbb{C}$ defined by gens x, y, z and relations (1)–(3)

LEM 54 $\exists$ Lie algebra iso $L \to \mathfrak{sl}_2$ that sends

$$x \longrightarrow \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$y \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$z \longrightarrow \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

pf One checks the 3 given matrices

satisfy (1) – (3).

So the given map $L \to sl_2$ is a Lie algebra

hom.

The map is surjective.

To show the map is injective, show $\dim L \le 3$

By const x, y, z gen L

the subspace

$$\text{Span}\{x, y, z\}$$

is closed under $[\ ,\ ]$ by (1) – (3), so

x, y, z span L. So $\dim L \le 3$.

Result follows. $\square$

A basis x, y, z for sl_2 is called <u>equitable</u> whenever

it satisfies (1) – (3)

Geometric meaning of equitable basis:

Let V = vector space over $\mathbb{C}$ with dim 2

View $sl_2 = \{u \in \mathrm{End}(V) \mid \mathrm{tr}\, u = 0\}$

Pick any 3 mutually distinct 1 dim'l subspaces

a, b, c of V.

V is the direct sum of any 2 of a, b, c

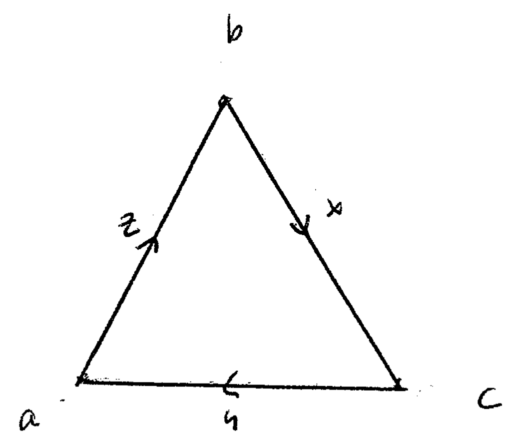
Using a, b, c we define 3 maps $x, y, z \in \mathrm{End}(V)$

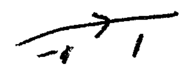
Each of x, y, z is diagonalizable, with eigenvalues

$1, -1$. The eigenspaces are:

gen	eigenspaces eigenvalue -1	eigenvalue 1
x	b	c
y	c	a
z	a	b

View





LEM 55 With above notation x, y, z is an equitable basis for sl_2

pf Pick non0 vectors

$u \in a, \quad v \in b, \quad w \in c$

u, v, w are lin dep, but any 2 are lin indep

so $\exists$ non0 $\alpha, \beta, \gamma \in \mathbb{C}$ s.t.

$$\alpha u + \beta v + \gamma w = 0$$

Replace
$$u \to \alpha u$$
$$v \to \beta v$$
$$w \to \gamma w$$

so
$$u + v + w = 0$$

Relative the basis u, w we see

$$u : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad v : \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad w : \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and

$$x : \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$y : \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$z : \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

Compare this with L54 to get result $\square$

Until further notice fix an equitable basis x, y, z for sl_2.

$\exists$ automorphism ρ of sl_2 that sends

$$x \to y \to z \to x$$

so $\rho^3 = I$.

Define a bilinear form

$$(\,,\,): \begin{array}{ccc} sl_2 \times sl_2 & \longrightarrow & \mathbb{C} \\ r \quad s & \longrightarrow & tr(rs) \end{array}$$

Then

$(\,,\,)$ is symmetric and nondegenerate

Also

$$([r,s],t) = (r,[s,t]) \qquad \forall r,s,t \in sl_2$$

For the equitable basis x, y, z of sl_2:

$(\,,\,)$	x	y	z
x	2	−2	−2
y	−2	2	−2
z	−2	−2	2

We now consider the basis for sl_2 that is dual to x, y, z wrt $(,)$

Define

$$x^* = -\frac{y+z}{2} \qquad y^* = -\frac{z+x}{2}$$

$$z^* = -\frac{x+y}{2}$$

LEM 56 For $r, s \in \{x, y, z\}$

$$(r, s^*) = 2\,\delta_{rs}$$

pf use the above table of inner products $\square$

LEM 57 Each of x^*, y^*, z^* is nilpotent

pf View

$$X = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \qquad y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad z = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}$$

so

$$X^* = -\frac{y+z}{2} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

$$y^* = -\frac{z+x}{2} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$z^* = -\frac{x+y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Each is nilp ✓ □

We now consider

$\exp(x^*)$, $\exp(y^*)$, $\exp(z^*)$

LEM 58

(i) $\exp(y^*)\, x\, \exp(-y^*) = -z$ (+ cp)

(ii) $\exp(y^*)\, z^*\, \exp(-y^*) = x^*$ (+ cp)

pf (i) View as 2×2 matrices

$$\exp(y^*)\, x \overset{?}{=} -z \exp(y^*)$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \overset{?}{=} -\begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad ✓ \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

(ii) Sim

□

LEM 59 the following coincide:

$$\exp(x^*)\exp(y^*),$$
$$\exp(y^*)\exp(z^*),$$
$$\exp(z^*)\exp(x^*).$$

pf

$$\exp(y^*)\exp(z^*)\exp(-y^*)$$
$$= \exp(y^*)(1+z^*)\exp(-y^*)$$
$$= 1 + \exp(y^*)\,z^*\exp(-y^*)$$
$$= 1 + x^*$$
$$= \exp(x^*)$$

So

$$\exp(x^*)\exp(y^*) = \exp(y^*)\exp(z^*)$$

Now apply ρ. $\square$

LEM 60 Let P denote the element in LEM 59. Then

$$\rho(r) = P r P^{-1} \qquad \forall r \in sl_2$$

pf

$$P x P^{-1} = \exp(x^*) \underbrace{\exp(y^*)\, x\, \exp(-y^*)}_{= -z} \exp(-x^*)$$

$$= - \underbrace{\exp(x^*)\, z\, \exp(-x^*)}_{= -y}$$

$$= y$$

By symmetry we also have

$P y P^{-1} = z, \qquad P z P^{-1} = x$ $\square$

LEM 61 We have

$$\rho = \exp(\operatorname{ad} x^*) \exp(\operatorname{ad} y^*)$$
$$= \exp(\operatorname{ad} y^*) \exp(\operatorname{ad} z^*)$$
$$= \exp(\operatorname{ad} z^*) \exp(\operatorname{ad} x^*)$$

pf Show

$$\rho = \exp(\operatorname{ad} x^*) \exp(\operatorname{ad} y^*)$$

$\forall r \in sl_2$ show

$$\rho(r) \overset{?}{=} \underbrace{\exp(\operatorname{ad} x^*) \underbrace{\exp(\operatorname{ad} y^*)(r)}_{= \exp(y^*)\, r \exp(-y^*)}}_{= \underbrace{\exp(x^*) \exp(y^*)}_{\rho}\, r\, \underbrace{\exp(-y^*) \exp(-x^*)}_{\rho^{-1}}}$$

$$\rho(r) = \rho r \rho^{-1}$$

ok.

$\square$

Now consider Lie algebra $\boxtimes$

Again let V denote a vector space over $\mathbb{C}$ with dim 2

LEM 62 Assume $r \in \mathrm{End}(V)$ is diagonalizable with eigenvalues $1, -1$.

Then $\forall s \in \mathrm{End}(V)$

$$[r,[r,[r,s]]] = 4[r,s] \qquad *$$

pf WLOG $V = \mathbb{C}^2$, $\mathrm{End}(V) = \mathrm{Mat}_2(\mathbb{C})$

$$r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h.$$

Obs $\mathrm{End}(V)$ has basis h, e, f, I

One verifies $*$ for $s \in \{h, e, f, I\}$ ✓ $\square$

Geometric meaning of $\boxtimes$

Pick any 4 mutually distinct 1-dim'l subspaces

a, b, c, d of V

V is direct sum of any 2 of a, b, c, d

Define a set

$$\mathbb{I} = \{a, b, c, d\}$$

For any distinct $i, j \in \mathbb{I}$ define

$$x_{ij} \in \text{End}(V)$$

such that

subspace i is an eigenspace for x_{ij} with eigenvalue -1
... j 1

Prop 63 The above maps

x_{ij} $\quad i,j \in \mathbb{I}$ $\quad i \neq j$

satisfy the defining relations for ⊠ from Prop 46.

pf Consider the relations (i) – (iii) in Prop 46.

(i) By constr ✓

(ii) By our discussion of the equitable basis for sl_2

(iii) By LEM 62

□

— o —

By Prop 63 V becomes a ⊠-module.