LECTURE 8 FRIDAY SEPT. 20

We continue to discuss the hypercube Q_D

Recall the set $\mathbb{I} = \{0,1,2,3\}$

In Lect 7, for distinct $i,j \in \mathbb{I}$ we defined an element $x_{ij} \in \mathfrak{sl}_2$.

Prop 46 We have

(i) For distinct $i,j \in \mathbb{I}$,

$$x_{ij} + x_{ji} = 0$$

(ii) For mutually distinct $i,j,k \in \mathbb{I}$,

$$[x_{ij}, x_{jk}] = 2x_{ij} + 2x_{jk}$$

(iii) For mutually distinct $i,j,k,l \in \mathbb{I}$

$$[x_{ij}, [x_{ij}, [x_{ij}, x_{kl}]]] = 4[x_{ij}, x_{kl}]$$

pf Routine check $\square$

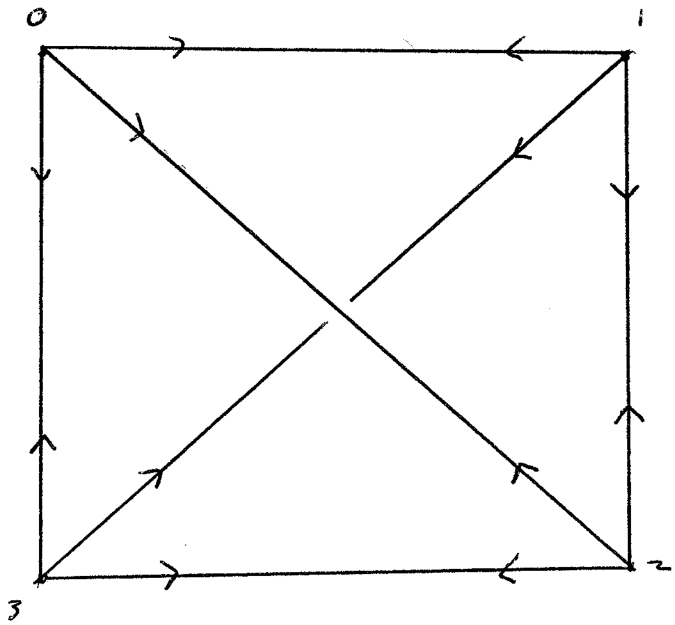
DEF 47 Let $\boxtimes$ denote the Lie algebra over $\mathbb{C}$ defined by generators

$$X_{ij} \qquad ij \in \mathbb{I} \qquad i \neq j$$

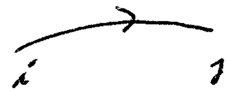
and relations (i) – (iii) in Prop 46.

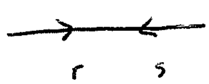
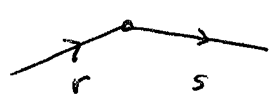
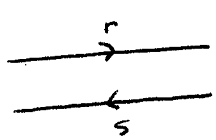
We call $\boxtimes$ the <u>tetrahedron</u> <u>algebra</u>.

Diagram



Each $\boxtimes$-generator X_{ij} is represented by a directed arc



picture	meaning
	$r+s=0$
	$[r,s]=2r+2s$
	$[r,[r,[r,s]]] = 4[r,s]$

By Prop 46 there exists a Lie algebra homomorphism

$$\boxtimes \longrightarrow \mathfrak{sl}_2 \qquad *$$

$$X_{ij} \longrightarrow x_{ij}$$

The standard module V for Φ_D supports an $\mathfrak{sl}_2$-module structure.

Pulling back the $\mathfrak{sl}_2$-action on V using $*$, V becomes a $\boxtimes$-module.

We now describe the $\boxtimes$ action on V.

LEM 48 For each generator of $\boxtimes$ we give the action on V in terms of A, A^*.

gen	acts on V as
x_{01}	$-A^*$
x_{10}	A^*
x_{12}	$A^* - A - \frac{[A,A^*]}{2}$
x_{21}	$A - A^* + \frac{[A,A^*]}{2}$
x_{23}	A
x_{32}	$-A$
x_{30}	$A^* - A + \frac{[A,A^*]}{2}$
x_{03}	$A - A^* - \frac{[A,A^*]}{2}$
x_{02}	$-A - A^* + \frac{[A,A^*]}{2}$
x_{20}	$A + A^* - \frac{[A,A^*]}{2}$
x_{13}	$A + A^* + \frac{[A,A^*]}{2}$
x_{31}	$-A - A^* - \frac{[A,A^*]}{2}$

pf On V, a acts as A and a^* acts as A^*. $\square$

LEM 49 For each generator of $\boxtimes$ we describe its action on V. We give 2 versions.

Action on V

gen	version I	version II
X_{01}	$-A^*$,	$S(-A)S^{-1}$
X_{10}	A^*	SAS^{-1}
X_{12}	$\exp(-R)A^*\exp(R)$,	$\exp(-R^*)(-A)\exp(R^*)$
X_{21}	$\exp(-R)(-A^*)\exp(R)$,	$\exp(-R^*)A\exp(R^*)$
X_{23}	SA^*S^{-1}	$-A$
X_{32}	$S(-A^*)S^{-1}$	$-A$
X_{30}	$\exp(L)A^*\exp(-L)$,	$\exp(L^*)(-A)\exp(-L^*)$
X_{03}	$\exp(L)(-A^*)\exp(-L)$,	$\exp(L^*)A\exp(-L^*)$
X_{02}	$\exp(-L)(-A^*)\exp(L)$,	$\exp(R^*)(-A)\exp(-R^*)$
X_{20}	$\exp(-L)A^*\exp(L)$,	$\exp(R^*)A\exp(-R^*)$
X_{13}	$\exp(R)A^*\exp(-R)$,	$\exp(-L^*)A\exp(L^*)$
X_{31}	$\exp(R)(-A^*)\exp(-R)$,	$\exp(-L^*)(-A)\exp(L^*)$

Pf Consider X_{12}

Show X_{12} acts on V as

$$\exp(-R)\, A^* \exp(R)$$

Recall that in sl_2,

$$X_{12} = \exp(-f)\, h \exp(f)$$

$$= \exp \operatorname{ad}(-f)\ (h)$$

$$= h - [f,h] + \frac{[f,[f,h]]}{2!} + \cdots$$

Recall $h = a^*$

On V_1

$$h = A^* \qquad f = R$$

So on V_1

$$X_{12} = A^* - [R, A^*] + \frac{[R,[R,A^*]]}{2!} + \cdots$$

$$= \exp(-R)\, A^* \exp(R).$$

The other assertions are similarly shown.

$\square$

LEM 50 Each $\boxtimes$ - generator x_{rs} is diagonalizable on V, with eigenvalues

$$D - 2i \qquad 0 \le i \le D \qquad *$$

For $0 \le i \le D$ the eigenspace for $D-2i$ has dimension

$$\binom{D}{i}$$

pf By construction A^* is diagonalizable with eigenvalues $*$. For $0 \le i \le D$ the eigenspace E_i^*V for A^* has dim $\binom{D}{i}$. Note that $-A^*$ also has eigenvalues $*$.

Now consider the $\boxtimes$ -generator x_{rs}. By LEM 49 $\exists$ invertible $B \in Mat_X(\mathbb{C})$ such that on V

$$B x_{rs} B^{-1} = \mp A^*$$

Therefore x_{rs} has the desired eigenvalues and eigenspace dimensions. $\square$

LEM 51 For each $\boxtimes$-generator X_{rs} and for $0 \le i \le D$, we describe the eigenspace of X_{rs} on V for the eigenvalue $D-2i$. We give 2 versions.

gen	eigenspace for the eigenvalue $D-2i$ version I	version II
X_{01}	$E^*_{D-i}V$	$S E_{D-i}V$
X_{10}	E^*_iV	$S E_iV$
X_{12}	$\exp(-R) E^*_iV$	$\exp(-R^*) E_{D-i}V$
X_{21}	$\exp(-R) E^*_{D-i}V$	$\exp(-R^*) E_iV$
X_{23}	$S E^*_iV$	E_iV
X_{32}	$S E^*_{D-i}V$	$E_{D-i}V$
X_{30}	$\exp(L) E^*_iV$	$\exp(L^*) E_{D-i}V$
X_{03}	$\exp(L) E^*_{D-i}V$	$\exp(L^*) E_iV$
X_{02}	$\exp(-L) E^*_{D-i}V$	$\exp(R^*) E_{D-i}V$
X_{20}	$\exp(-L) E^*_iV$	$\exp(R^*) E_iV$
X_{13}	$\exp(R) E^*_iV$	$\exp(-L^*) E_iV$
X_{31}	$\exp(R) E^*_{D-i}V$	$\exp(-L^*) E_{D-i}V$

pf Use LEM 49

Consider X_{12}.

We show that for the eigenvalue $\theta - 2i$

the corresponding eigenspace is

$$\exp(-R)\, E_i^* V$$

Abbr
$$B = \exp(-R)$$

By LEM 49 X_{12} acts on V as

$$BA^* B^{-1}$$

For $\theta - 2i$ the eigenspace for A^* is

$$E_i^* V$$

So for $\theta - 2i$ the eigenspace for BA^*B^{-1} is

$$B E_i^* V,$$

as desired. The remaining assertions are similarly shown. $\square$

LEM 52 For $0 \le i \le D$ and each $\boxtimes$-generator x_{rs} we display some bases for the eigenspace on V for the eigenvalue $D-2i$.

gen	bases for the eigenspace on V with eigval $D-2i$
X_{01}	$\{\hat{y}\}_{y \in \Gamma_{D-i}(x)}$
X_{10}	$\{\hat{y}\}_{y \in \Gamma_i(x)}$
X_{12}	$\{\exp(-R)\hat{y}\}_{y \in \Gamma_i(x)}$, $\{\exp(-R^*)\check{y}\}_{y \in \Gamma_{D-i}(x)}$
X_{21}	$\{\exp(-R)\hat{y}\}_{y \in \Gamma_{D-i}(x)}$ $\{\exp(-R^*)\check{y}\}_{y \in \Gamma_i(x)}$
X_{23}	$\{\check{y}\}_{y \in \Gamma_i(x)}$
X_{32}	$\{\check{y}\}_{y \in \Gamma_{D-i}(x)}$
X_{30}	$\{\exp(L)\hat{y}\}_{y \in \Gamma_i(x)}$, $\{\exp(L^*)\check{y}\}_{y \in \Gamma_{D-i}(x)}$
X_{03}	$\{\exp(L)\hat{y}\}_{y \in \Gamma_{D-i}(x)}$ $\{\exp(L^*)\check{y}\}_{y \in \Gamma_i(x)}$
X_{02}	$\{\exp(-L)\hat{y}\}_{y \in \Gamma_{D-i}(x)}$ $\{\exp(R^*)\check{y}\}_{y \in \Gamma_{D-i}(x)}$
X_{20}	$\{\exp(-L)\hat{y}\}_{y \in \Gamma_i(x)}$ $\{\exp(R^*)\check{y}\}_{y \in \Gamma_i(x)}$
X_{13}	$\{\exp(R)\hat{y}\}_{y \in \Gamma_i(x)}$ $\{\exp(-L^*)\check{y}\}_{y \in \Gamma_i(x)}$
X_{31}	$\{\exp(R)\hat{y}\}_{y \in \Gamma_{D-i}(x)}$ $\{\exp(-L^*)\check{y}\}_{y \in \Gamma_{D-i}(x)}$

pf Use LEM 51.

Recall that

$$\{\hat{y} \mid y \in \Gamma_i(x)\}$$

is a basis for E_i^*V, and

$$\{\check{y} \mid y \in \Gamma_i(x)\}$$

is a basis for E_iV. $\square$

THM 53 For $0 \le i \le D$ and each generator of $\boxtimes$ we display the eigenspace on V for the eigenvalue $D-2i$

gen	eigenspace for the eigenvalue $D-2i$
x_{01}	$E^*_{D-i}V$
x_{10}	$E^*_i V$
x_{12}	$(E^*_0 V + \cdots + E^*_i V) \cap (E_0 V + \cdots + E_{D-i} V)$
x_{21}	$(E^*_D V + \cdots + E^*_{D-i} V) \cap (E_0 V + \cdots + E_i V)$
x_{23}	$E_i V$
x_{32}	$E_{D-i} V$
x_{30}	$(E^*_0 V + \cdots + E^*_i) \cap (E_0 V + \cdots + E_{D-i} V)$
x_{03}	$(E^*_0 V + \cdots + E^*_{D-i}) \cap (E_0 V + \cdots + E_i V)$
x_{02}	$(E^*_0 V + \cdots + E^*_{D-i}) \cap (E_D V + \cdots + E_{D-i} V)$
x_{20}	$(E^*_0 V + \cdots + E^*_i) \cap (E_D V + \cdots + E_i V)$
x_{13}	$(E^*_D V + \cdots + E^*_i) \cap (E_0 V + \cdots + E_i V)$
x_{31}	$(E^*_D V + \cdots + E^*_{D-i}) \cap (E_0 V + \cdots + E_{D-i} V)$

pf Use LEM 51.

Consider X_{12}

let U_i = eigenspace for X_{12} on V, associated with the eigenvalue $D-2i$.

By LEM 51.

$$U_i = \exp(-R)E_i^*V$$

$$\subseteq E_i^*V + RE_i^*V + R^2E_i^*V + \cdots$$

$$\subseteq E_i^*V + E_{i+1}^*V + \cdots + E_D^*V$$

Obs

$$U_i + U_{i+1} + \cdots + U_D \subseteq E_i^*V + E_{i+1}^*V + \cdots + E_D^*V \qquad (*)$$

On each side of $(*)$ the sum is direct.
On each side of $(*)$ the dimensions are the same.
So equality holds in $(*)$. Thus

$$U_i + \cdots + U_D = E_i^*V + \cdots + E_D^*V$$

By LEM 51,

$$U_i = \exp(-R^*) E_{D-i}V$$
$$\subseteq E_{D-i}V + R^* E_{D-i}V + (R^*)^2 E_{D-i}V + \cdots$$
$$\subseteq E_{D-i}V + E_{D-i+1}V + \cdots + E_D V$$

Obs

$$U_0 + U_1 + \cdots + U_i \subseteq E_{D-i}V + E_{D-i+1}V + \cdots + E_D V \qquad (\star)$$

On each side of $(\star)$ the sum is direct.

On each side of $(\star)$ the dimensions are the same.

So equality holds in $(\star)$. Thus

$$U_0 + \cdots + U_i = E_{D-i}V + \cdots + E_D V$$

Now

$$U_i = (U_i + \cdots + U_D) \cap (U_0 + \cdots + U_i)$$

$$= (E_i^* V + \cdots + E_D^* V) \cap (E_{D-i}V + \cdots + E_D V)$$

The remaining assertions are similarly shown. $\square$

the above results on $\boxtimes$ and Q_D

are due to 2 groups of researchers.

Suogang Gao, Bo Hou,

obtained the $\boxtimes$ action on the standard

module V of Q_D.

Arlene Pascasio and Randy Penaflor

established the connection between the $\boxtimes$ action

and sl_2.