LECTURE 7 WEDNESDAY SEPT. 18

We continue to discuss the hypercube Q_D

Motivated by LEM 35 and $a^* = h$, define

$$e^* = \frac{[a, a^*] + 2a^*}{4} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$f^* = \frac{[a^*, a] + 2a^*}{4} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$h^* = a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Obs

$$e^* + f^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = a^*$$

One checks

$$e^*, f^*, h^*$$

is a basis for sl_2, and

$$[e^*, f^*] = h^*$$

$$[h^*, e^*] = 2e^*$$

$$[h^*, f^*] = -2f^*$$

LEM 39. On the sl_2-module V

generator	e^*	f^*	h^*
action on V	L^*	R^*	A

pf Compare the definitions of e^*, f^* with LEM 35. $\square$

Notational aside

Recall standard module V for Q_D

Pick $y \in X$ and write

$$y = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_D \qquad \varepsilon_i \in \{1, -1\}$$

View $\{1, -1\}$ as vertex set for 1-cube Q_1

Let $\mathbb{V}$ denote the standard module for Q_1

with basis $\hat{1}, \quad \widehat{-1}$

Identify

$$V = \underbrace{\mathbb{V} \otimes \mathbb{V} \otimes \cdots \otimes \mathbb{V}}_{D} \qquad \otimes = \otimes_{\mathbb{C}}$$

via

$$\hat{y} = \hat{\varepsilon}_1 \otimes \hat{\varepsilon}_2 \otimes \cdots \otimes \hat{\varepsilon}_D$$

Earlier we turned V into an sl_2-module.

In this construction we set $D=1$ and use the base vertex "1" to turn $\mathbb{V}$ into an sl_2-module.

From this point of view

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{is adjacency matrix of } Q_1$$

$$a^* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{is dual adj matrix of } Q_1 \text{ wrt } 1$$

e = lowering matrix for Q_1 wrt 1

f = raising matrix --

e^* = dual lowering --

f^* = dual raising --

Define

$$\check{1} = \frac{\hat{1} + \widehat{-1}}{\sqrt{2}}$$

$$\check{-1} = \frac{\hat{1} - \widehat{-1}}{\sqrt{2}}$$

$\check{1}$ is an eigenvector for a with eigenvalue 1

$\check{-1}$ - - - - -1

Define $A \in \mathrm{End}(\mathbb{V})$ s.t.

$$A \hat{\varepsilon} = \check{\varepsilon} \qquad \text{for } \varepsilon \in \{1, -1\}$$

With respect to the basis $\hat{1}$ $\hat{-1}$

$$A : \quad \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

So

$$A^2 = I$$

and

$$A \check{\varepsilon} = \hat{\varepsilon} \qquad \text{for } \varepsilon \in \{1, -1\}$$

Obs
$$e^* \check{1} = 0, \qquad e^* \check{-1} = 1$$
$$f^* \check{1} = \check{-1} \qquad f^* \check{-1} = 0$$

Let W, W' denote sl_2-modules. Then $W \otimes W'$ is an sl_2-module with

$$z.(w \otimes w') = (z.w) \otimes w' + w \otimes (z.w')$$
$$\forall z \in sl_2, \quad w \in W, \quad w' \in W'$$

Then

$$V = \mathbb{V} \otimes \mathbb{V} \otimes \cdots \otimes \mathbb{V} \qquad \text{as } sl_2\text{-modules}$$

Given $y \in X$ write

$$y = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_D \qquad \varepsilon_i \in \{1, -1\}$$

Define

$$\check{y} = \check{\varepsilon}_1 \otimes \check{\varepsilon}_2 \otimes \cdots \otimes \check{\varepsilon}_D$$

One checks

$$\{\check{y} \mid y \in X\} \text{ is orthonormal basis for } V$$

Define $S \in \mathrm{Mat}_X(\mathbb{C})$ s.t.

$$S \hat{y} = \check{y} \qquad \forall y \in X$$

One checks

$$S\, u_1 \otimes u_2 \otimes \cdots \otimes u_D = s u_1 \otimes s u_2 \otimes \cdots \otimes s u_D \qquad u_i \in \mathbb{V},\ 1 \le i \le D$$

$$\bar{S} = S, \qquad S^t = S$$

$$S^2 = I$$

$$S A S^{-1} = A^*$$

$$S A^* S^{-1} = A$$

$$S E_i S^{-1} = E_i^* \qquad 0 \le i \le D$$

$$S E_i^* S^{-1} = E_i$$

$$S R S^{-1} = R^*$$ $$S R^* S^{-1} = R$$

$$S L S^{-1} = L^*$$ $$S L^* S^{-1} = L$$

Moreover for $0 \le i \le D$ the following is an orthonormal basis for $E_i V$:

$$\left\{ \check{y} \;\middle|\; y \in \Gamma_i(x) \right\}$$

ex For $y \in X$

$$\check{y} = 2^{-D/2} \sum_{z \in X} \hat{z} (-1)^{\frac{\partial(x,y) + \partial(x,z) - \partial(y,z)}{2}}$$

Comments on $\mathfrak{sl}_2$

For $u \in \mathfrak{sl}_2$ define the $\mathbb{C}$-linear map

$$\operatorname{ad} u : \begin{array}{ccc} \mathfrak{sl}_2 & \longrightarrow & \mathfrak{sl}_2 \\ v & \longrightarrow & [u,v] \end{array}$$

We recall the exponential map.

For any finite dim'l vector space W and $z \in \operatorname{End}(W)$

z is called nilpotent whenever $\exists\ n \geq 1$ such that $z^n = 0$

Assume that W is an $\mathfrak{sl}_2$-module. Then for $z \in \mathfrak{sl}_2$

if z is nilpotent on W then

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is a well defined element of $\operatorname{End}(W)$.

Assume z is nilpotent. One checks that on W

$$\exp(z)\exp(-z) = I.$$

Also for $u \in \mathfrak{sl}_2$ the following holds on W:

$$\begin{aligned} \exp(z)\, u \exp(-z) &= \exp(\operatorname{ad} z)(u) \\ &= u + [z,u] + \frac{[z,[z,u]]}{2!} + \cdots \end{aligned}$$

Back to φ_0

Recall

$$L E_i^* V \subseteq E_{i-1}^* V, \qquad R E_i^* V \subseteq E_{i+1}^* V \qquad 0 \le i \le D$$

So

$$L^{D+1} = 0, \qquad R^{D+1} = 0$$

So

$$L, \quad R \quad \text{nilpotent on } V$$

Consider

$$\exp(tL), \qquad \exp(tR) \qquad t \in \mathbb{C}$$

Define a partial order $\le$ on X as follows:

$\forall y, z \in X$

$$y \le z \quad \text{whenever} \quad \partial(x,y) + \partial(y,z) = \partial(x,z)$$

LEM 40 For Q_0, pick $y \in X$ and write

$i = \partial(x, y)$.

(i) For $0 \le n \le i$

$$L^n \hat{y} = n! \sum_{\substack{z \le y \\ \partial(y,z)=n}} \hat{z}$$

(ii) For $n > i$

$$L^n \hat{y} = 0$$

(iii) For $0 \le n \le D-i$

$$R^n \hat{y} = n! \sum_{\substack{y \le z \\ \partial(y,z)=n}} \hat{z}$$

(iv) For $n > D-i$

$$R^n \hat{y} = 0$$

pf Recall $c_j = j$ for $0 \le j \le D$ $\square$

COR 41 For $t \in \mathbb{C}$ and $y \in X$

$$\exp(tL)\,\hat{y} = \sum_{z \le y} t^{\partial(y,z)}\,\hat{z}$$

$$\exp(tR)\,\hat{y} = \sum_{y \le z} t^{\partial(y,z)}\,\hat{z}$$

pf By L40 and the def of exp $\square$

— o —

We also have

$$(L^*)^{D+1} = 0, \qquad (R^*)^{D+1} = 0$$

Consider

$$\exp(tL^*), \qquad \exp(tR^*) \qquad t \in \mathbb{C}$$

LEM 42 For Φ_D, pick $y \in X$ and write $i = \partial(x,y)$.

(i) For $0 \le n \le i$

$$(L^*)^n \check{y} = n! \sum_{\substack{z \le y \\ \partial(y,z)=n}} \check{z}$$

(ii) For $n > i$

$$(L^*)^n \check{y} = 0$$

(iii) For $0 \le n \le D - i$

$$(R^*)^n \check{y} = n! \sum_{\substack{z \ge y \\ \partial(y,z)=n}} \check{z}$$

(iv) For $n > D - i$

$$(R^*)^n \check{y} = 0$$

pf In L40 Apply S to each side and use

$$SL = L^* S, \qquad SR = R^* S, \qquad S\hat{y} = \check{y}$$

$\square$

COR 43 For $t \in \mathbb{C}$ and $y \in X$

$$\exp(tL^*)\check{y} = \sum_{z \le y} t^{\partial(y,z)} \check{z}$$

$$\exp(tR^*)\check{y} = \sum_{y \le z} t^{\partial(y,z)} \check{z}$$

pf Use LEM 42 and def of exp $\square$

— o —

In order to better understand

$\exp(tL)$, $\exp(tR)$, $\exp(tL^*)$, $\exp(tR^*)$

Consider

$\exp(te)$, $\exp(tf)$, $\exp(te^*)$, $\exp(tf^*)$

LEM 44

z	$\exp(tz)$
e	$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$
f	$\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$
e^*	$\frac{1}{2}\begin{pmatrix} t+2 & -t \\ t & 2-t \end{pmatrix}$
f^*	$\frac{1}{2}\begin{pmatrix} 2+t & t \\ -t & 2-t \end{pmatrix}$

pf $\quad z^2 = 0 \quad$ so

$$\exp(tz) = I + tz$$

LEM 45 For $t \in \mathbb{C}$ we have

$$\exp(te)\, h \exp(-te) = \begin{pmatrix} 1 & -2t \\ 0 & -1 \end{pmatrix}$$

$$\exp(tf)\, h \exp(-tf) = \begin{pmatrix} 1 & 0 \\ 2t & -1 \end{pmatrix}$$

$$\exp(te^*)\, h^* \exp(-te^*) = \begin{pmatrix} -t & 1+t \\ 1-t & t \end{pmatrix}$$

$$\exp(tf^*)\, h^* \exp(-tf^*) = \begin{pmatrix} t & 1+t \\ 1-t & -t \end{pmatrix}$$

pf Routine $\square$

In above Lemma we now take $t = \mp 1$

Let $\mathbb{I}$ denote a set with $|\mathbb{I}| = 4$

Take $\mathbb{I} = \{0, 1, 2, 3\}$

For all distinct $i, j \in \mathbb{I}$ we define

$x_{ij} \in sl_2$ as follows

$$X_{01} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -h = -a^*$$

$$X_{12} = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix} = \exp(-f)\, h \exp(f)$$

$$= \exp(-f^*)(-h^*) \exp(f^*)$$

$$= a^* - a - \frac{[a, a^*]}{2}$$

$$X_{23} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = h^* = a$$

$$X_{30} = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix} = \exp(e)\, h \exp(-e)$$

$$= \exp(e^*)(-h^*) \exp(-e^*)$$

$$= a^* - a + \frac{[a, a^*]}{2}$$

$$X_{02} = \begin{pmatrix} -1 & -2 \\ 0 & 1 \end{pmatrix} = \exp(-e)(-h)\exp(e)$$

$$= \exp(f^*)(-h^*)\exp(-f^*)$$

$$= -a - a^* + \frac{[a, a^*]}{2}$$

$$X_{13} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \exp(f)\, h\, \exp(-f)$$

$$= \exp(-e^*)\, h^*\, \exp(e^*)$$

$$= a + a^* + \frac{[a, a^*]}{2}$$

All remaining X_{ij} defined by

$$X_{ij} = -X_{ji}$$