LECTURE 6 MONDAY SEPT. 16

We now consider a family of distance-regular graphs called the hypercubes.

DEF 2.6 Fix an integer $D \geq 1$. The graph Q_D has vertex set

$$X = \text{set of all sequences } \varepsilon_1 \varepsilon_2 \cdots \varepsilon_D \text{ such that}$$

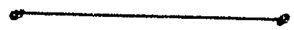
$$\varepsilon_i \in \{1, -1\} \quad \text{for } 1 \leq i \leq D$$

and edge set

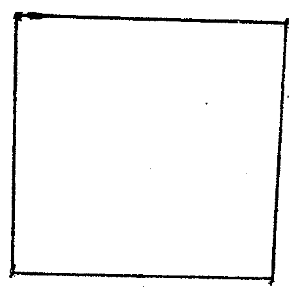
$$R = \{ xy \mid x, y \in X, \; x, y \text{ differ in exactly one coordinate} \}$$

Call Q_D the <u>D-cube</u> or <u>hypercube</u>. Q_D is also called the Hamming graph $H(D, 2)$.

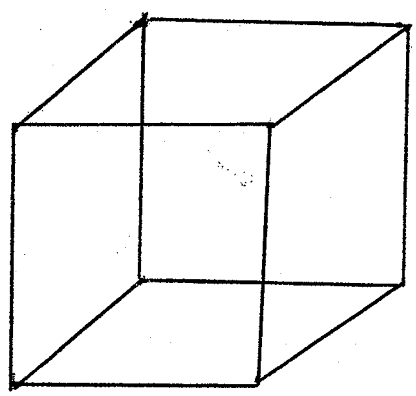
Q_1



Q_2



Q_3



Basic facts about Φ_0.

- The graph Φ_0 is connected. For vertices x, y of Φ_0 the distance

$$\partial(x,y) = \#\text{ of coordinates at which } x, y \text{ differ.}$$

Moreover the diameter of Φ_0 is D.

- The graph Φ_0 is bipartite. The bipartition is

$$X = X^+ \cup X^-$$

$$X^+ = \{ x \in X \mid x \text{ has even weight} \}$$

$$X^- = \{ x \in X \mid x \text{ has odd weight} \}$$

where the weight of x is the number of positive coords of x

- The graph Φ_0 is distance-regular, with intersection numbers

$$c_i = i \qquad a_i = 0 \qquad b_i = D - i \qquad 0 \le i \le D$$

the valencies are

$$k_i = \binom{D}{i} \qquad 0 \le i \le D$$

Consider the eigenvalues of φ_D.

Find pattern

φ_1

eigval	1	-1
mult	1	1

φ_2

eigval	2	0	-2
mult	1	2	1

φ_3

eigval	3	1	-1	-3
mult	1	3	3	1

Not hard to guess

ex The eigenvalues of φ_D are

$$\theta_i = D - 2i \qquad 0 \le i \le D$$

with θ_i having multiplicity $\binom{D}{i}$

for $0 \le i \le D$.

Until further notice fix $x \in X$

and write $M^* = M^*(x)$, $T = T(x)$, etc.

WLOG $x = \underbrace{11\cdots 1}_{D}$

DEF 27 For Q_0 Define $R, L \in \mathrm{Mat}_X(\mathbb{C})$ by

$$R = \sum_{i=0}^{D-1} E^*_{i+1} A E^*_i \qquad \text{"raising matrix"}$$

$$L = \sum_{i=1}^{D} E^*_{i-1} A E^*_i \qquad \text{"lowering matrix"}$$

We observe

$$\bar{R} = R, \qquad \bar{L} = L, \qquad R^t = L$$

$$A = R + L$$

$$R E^*_i V \subseteq E^*_{i+1} V \qquad 0 \le i \le D$$

$$L E^*_i V \subseteq E^*_{i-1} V \qquad 0 \le i \le D$$

DEF 28 For Q_D Define

$$A^* = \sum_{i=0}^{D} \theta_i^* E_i^*$$

where

$$\theta_i^* = D-2i \qquad 0 \le i \le D.$$

— o —

We will show that A^* is a dual adjacency matrix wrt x and $\{\theta_i^*\}_{i=0}^{D}$

LEM 29 For Q_D pick any $y, z \in X$ such that

$\partial(y,z) = 2$ and $\partial(x,y) = \partial(x,z)$.

(i) $\exists$ unique vertex $u \in X$ s.t

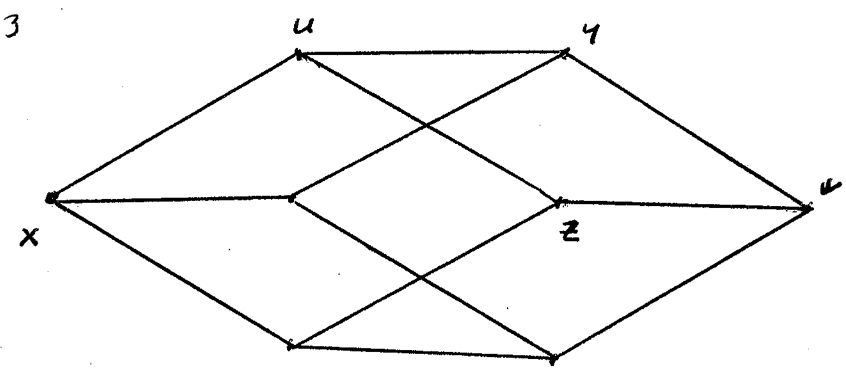
$\partial(u,y) = 1, \quad \partial(u,z) = 1, \quad \partial(u,x) = \partial(x,y) - 1$

(ii) $\exists$ unique vertex $v \in X$ s.t.

$\partial(v,y) = 1, \quad \partial(v,z) = 1, \quad \partial(v,x) = \partial(x,y) + 1$

pf Routine

illustration for $D=3$



LEM 30 We have

(i) $LR - RL = A^*$

(ii) $A^*L - LA^* = 2L$

(iii) $A^*R - RA^* = -2R$

pf (i) For $y, z \in X$ we compute the (y,z)-entry of each side.

Case	$(LR)_{yz}$	$(RL)_{yz}$	$(A^*)_{yz}$
$y = z \in \Gamma_i(x)$	b_i	c_i	$D-2i$
$\partial(y,z) = 2$ $\partial(x,y) = \partial(x,z)$	1	1	0
otherwise	0	0	0

Result follows.

(ii), (iii) Similar

□

LEM 31

(i) $$R = \frac{AA^* - A^*A + 2A}{4}$$

(ii) $$L = \frac{A^*A - AA^* + 2A}{4}$$

pf Eliminate A using $A = R + L$

and simplify using L30. $\square$

Prop 32 We have

(i) $$A^2A^* - 2AA^*A + A^*A^2 = 4A^*$$

(ii) $$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A$$

pf (i)

$[u,v] = uv - vu$

$$\begin{aligned}
LHS &= [A, [A, A^*]] \\
&= [\underbrace{A}_{= R+L}, \underbrace{[R+L, A^*]}_{= 2R-2L}] \\
&= 2\,[R+L, R-L] \\
&= 2\,\underbrace{[L,R]}_{= A^*} - 2\,\underbrace{[R,L]}_{= -A^*} \\
&= 4A^*
\end{aligned}$$

(ii) Similar. □

Recall the primitive idempotent E_i for $\theta_i = D - 2i$ $(0 \le i \le D)$.

Prop 33 A^* is a dual adjacency matrix wrt x and $\{\theta_i\}_{i=0}^{D}$. Moreover

$$E_i A^* E_i = 0 \qquad 0 \le i \le D$$

pf We show

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j| \ne 1 \qquad (0 \le i,j \le D)$$

Let i,j be given. By Prop 32 (i)

$$0 = E_i \left(A^2 A^* - 2AA^*A + A^*A^2 - 4A^* \right) E_j$$

$$= E_i A^* E_j \Big(\underbrace{\theta_i^2 - 2\theta_i\theta_j + \theta_j^2 - 4}_{\begin{gathered} \shortparallel \\ (\theta_i - \theta_j)^2 - 4 \\ \shortparallel \\ (\theta_i - \theta_j - 2)(\theta_i - \theta_j + 2) \\ \shortparallel \\ 4(i-j+1)(i-j-1) \end{gathered}} \Big)$$

So $E_i A^* E_j = 0$ if $|i-j| \ne 1$ $\square$

Earlier we defined the rasing matrix R and lowering matrix L.

We now define dual versions R^* and L^*

DEF 34 For Φ define

$$R^* = \sum_{i=0}^{D-1} E_{i+1} A^* E_i$$

$$L^* = \sum_{i=1}^{D} E_{i-1} A^* E_i$$

We obs

$$\overline{R^*} = R^*, \qquad \overline{L^*} = L^*, \qquad (R^*)^t = L^*$$

$$A^* = R^* + L^*$$

$$R^* E_i V \subseteq E_{i+1} V \qquad 0 \le i \le D$$

$$L^* E_i V \subseteq E_{i-1} V \qquad 0 \le i \le D$$

LEM 35 We have

(i) $$R^* = \frac{A^*A - AA^* + 2A^*}{4}$$

(ii) $$L^* = \frac{AA^* - A^*A + 2A^*}{4}$$

pf. (i) Write

$$C = \frac{A^*A - AA^* + 2A^*}{4}$$

View

$$C = I C I \qquad I = \sum_{i=0}^{D} E_i$$

$$= \sum_{0 \le i,j \le D} E_i C E_j$$

$$= \sum_{0 \le i,j \le D} \underbrace{E_i A^* E_j}_{\substack{\parallel \\ 0 \text{ if } |i-j| \ne 1}} \underbrace{\frac{\theta_j - \theta_i + 2}{4}}_{\substack{\parallel \\ \begin{cases} 0 & \text{if } i-j=-1 \\ 1 & \text{if } i-j=1 \end{cases}}}$$

$$= R^*$$

(ii) Similar

$\square$

LEM 36 We have

(i) $L^*R^* - R^*L^* = A$

(ii) $AL^* - L^*A = 2L^*$

(iii) $AR^* - R^*A = -2R^*$

pf To verify each equation, elim L^*, R^* using L35 and evaluate the result using Prop 32. $\square$

Recall the Lie algebra $sl_2 = sl_2(\mathbb{C})$ is the $\mathbb{C}$-vector space consisting of the 2×2 matrices / $\mathbb{C}$ that have trace 0, together with the Lie bracket

$$[u, v] = uv - vu \qquad u, v \in sl_2$$

sl_2 has a basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$[e, f] = h$$

$$[h, e] = 2e$$

$$[h, f] = -2f$$

DEF 37 The <u>universal enveloping algebra</u> $U(sl_2)$ is the (associative) $\mathbb{C}$-algebra with 1 defined by generators E, F, H and relations

$$EF - FE = H$$

$$HE - EH = 2E$$

$$HF - FH = -2F$$

Thm 38. For q_0 $\exists$ surjective $\mathbb{C}$-alg homomorphism

$$U(sl_2) \to T$$

that sends

$$E \to L$$

$$F \to R$$

$$H \to A^*$$

By thm 38 the standard module V becomes an $\mathfrak{sl}_2$-module such that

generator	e	f	h
action on V	L	R	A^*

— o —

Define

$$a = e + f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$a^* = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

On the $\mathfrak{sl}_2$-module V

generator	a	a^*
action on V	A	A^*

One checks

$$[a, [a, a^*]] = 4a^* \qquad \text{(compare with Prop 32)}$$

$$[a^*, [a^*, a]] = 4a$$