LECTURE 5 FRIDAY SEPT. 13

Recall our connected graph $\Gamma = (X, \mathcal{R})$, $|X| \geq 2$

Fix $x \in X$, write $M^* = M^*(x)$, $T = T(x)$, $d = \rho_x$

Until further notice assume that Γ is distance-regular with respect to x. Obs

$$c_i \neq 0 \qquad 1 \leq i \leq d, \qquad c_0 = 0$$

$$b_i \neq 0 \qquad 0 \leq i \leq d-1, \qquad b_d = 0$$

$$a_0 = 0, \qquad c_1 = 1$$

LEM 13

(i) $\quad k_i c_i = k_{i-1} b_{i-1} \qquad 1 \leq i \leq d$

(ii) $\quad k_i = \dfrac{b_0 b_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i} \qquad 0 \leq i \leq d$

pf (i) Count in two ways the edges between $\Gamma_{i-1}(x)$ and $\Gamma_i(x)$.

(ii) By (i) and induction on i. $\square$

For $0 \le i \le d$ define

$$\mathbb{1}_i = E_i^* \mathbb{1} = \sum_{y \in \Gamma_i(x)} \hat{y}$$

So

$$\mathbb{1}_0 = \hat{x},$$

$$\langle \mathbb{1}_i, \mathbb{1}_j \rangle = \delta_{ij} k_i \qquad 0 \le i, j \le d$$

$$E_i^* \mathbb{1}_j = \delta_{ij} \mathbb{1}_j \qquad 0 \le i, j \le d$$

We now consider the action of A on $\{\mathbb{1}_i\}_{i=0}^{d}$.

LEM 14

(i) $A \mathbb{1}_0 = \mathbb{1}_1$

(ii) $A \mathbb{1}_i = b_{i-1} \mathbb{1}_{i-1} + a_i \mathbb{1}_i + c_{i+1} \mathbb{1}_{i+1} \qquad 1 \le i \le d-1$

(iii) $A \mathbb{1}_d = b_{d-1} \mathbb{1}_{d-1} + a_d \mathbb{1}_d$

pf (ii) For $1 \le i \le d-1$,

$$A \mathbb{1}_i = A \sum_{y \in \Gamma_i(x)} \hat{y}$$

$$= \sum_{y \in \Gamma_i(x)} \sum_{z \in \Gamma(y)} \hat{z}$$

$$= \sum_{z \in X} \hat{z} \, |\Gamma(z) \cap \Gamma_i(x)|$$

$$= b_{i-1} \sum_{z \in \Gamma_{i-1}(x)} \hat{z} + a_i \sum_{z \in \Gamma_i(x)} \hat{z}$$

$$+ c_{i+1} \sum_{z \in \Gamma_{i+1}(x)} \hat{z}$$

$$= b_{i-1} \mathbb{1}_{i-1} + a_i \mathbb{1}_i + c_{i+1} \mathbb{1}_{i+1}$$

(i), (iii) Similar. $\square$

LEM 15 The vectors $\{\mathbb{1}_i\}_{i=0}^d$ form a basis for the primary T-module. Relative this basis,

$$A: \begin{pmatrix} a_0 & b_0 & & & \mathbf{0} \\ c_1 & a_1 & b_1 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{d-1} \\ \mathbf{0} & & & c_d & a_d \end{pmatrix}$$

$$E_i^*: \quad \underset{\substack{\uparrow \\ i \text{ coord}}}{\mathrm{diag}(0, \ldots, 0, 1, 0, \ldots, 0)} \qquad 0 \le i \le d.$$

pf Let W denote the subspace of V spanned by $\{\mathbb{1}_i\}_{i=0}^d$. By LEM 14, $AW \subseteq W$.

We saw $E_i^* \mathbb{1}_j = \delta_{ij} \mathbb{1}_j$ $(0 \le i, j \le d)$ so $E_i^* W \subseteq W$ for $0 \le i \le d$. So W is a T-module.

Let $\tilde{W}$ denote the primary T-module. Show $W = \tilde{W}$.

By construction $\hat{x} = \mathbb{1}_0 \in W$. Also $\hat{x} \in \tilde{W}$

So $W \cap \tilde{W} \neq 0$.

$W \cap \tilde{W}$ is a nonO T-module contained in $\tilde{W}$. T-module $\tilde{W}$ is irred, so $W \cap \tilde{W} = \tilde{W}$, ie $\tilde{W} \subseteq W$

By construction

$$\dim W = d+1$$

We saw earlier

$$\dim \tilde{W} \geq d+1.$$

So $W = \tilde{W}$. $\square$

We now bring in some polynomials in one variable.

Let λ = indeterminate.

Let $\mathbb{C}[\lambda]$ = $\mathbb{C}$-algebra of polynomials in λ that have all coefs in $\mathbb{C}$.

For $0 \le i \le d+1$ define $f_i \in \mathbb{C}[\lambda]$ by

$$f_0 = 1, \qquad f_1 = \lambda$$

$$\lambda f_i = b_{i-1} f_{i-1} + a_i f_i + c_{i+1} f_{i+1} \qquad 1 \le i \le d-1$$

$$\lambda f_d = b_{d-1} f_{d-1} + a_d f_d + \frac{f_{d+1}}{c_1 c_2 \cdots c_d}$$

Observe that for $0 \le i \le d$

f_i has degree i, and coef of λ^i is $\dfrac{1}{c_1 c_2 \cdots c_i}$

Also

f_{d+1} is monic with degree $d+1$.

LEM 16

(i) $f_i(A)\hat{x} = \mathbb{1}_i \qquad 0 \le i \le d$

(ii) $f_{d+1}(A)\hat{x} = 0$

pf Compare the def of f_i with LEM 14. $\square$

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Recall that M is the subalgebra of $Mat_X(\mathbb{C})$ generated by A

LEM 17

(i) the primary T-module is $M\hat{x}$

(ii) For the action of A on $M\hat{x}$, f_{d+1} is both the min poly and char poly.

pf (i) By LEM 16 $M\hat{x}$ has basis $\{\mathbb{1}_i\}_{i=0}^{d}$. Done by LEM 15.

(ii) $f_{d+1}(A) M\hat{x} = M f_{d+1}(A)\hat{x} = 0.$

So min poly of A on $M\hat{x}$ divides f_{d+1}

A is diagonalizable on $M\hat{x}$ so on $M\hat{x}$

min poly of A = char poly of A.

char poly of A on $M\hat{x}$ has degree $d+1$

f_{d+1} has degree $d+1$

Result follows. $\square$

DEF 18 Let $\{\theta_i\}_{i=0}^{d}$ denote the roots of f_{d+1}.

[these roots are among the eigenvalues of Γ]

Call $\{\theta_i\}_{i=0}^{d}$ the <u>primary</u> <u>eigenvalues</u> <u>of Γ</u> <u>with respect to x</u>

LEM 19 For any eigenvalue θ_i of Γ,

(i) Assume θ_i is primary. Then $E_i\hat{x}$ is a basis for $E_iM\hat{x}$

(ii) Assume θ_i is not primary. Then $E_i\hat{x} = 0$

pf (i) By def $E_iM\hat{x} \neq 0$. E_i is a prim idempotent

so $E_iM = \mathbb{C}E_i$.

So $E_iM\hat{x} = \mathbb{C}E_i\hat{x} \neq 0$

(ii) By def of primary, $E_iM\hat{x} = 0$

$\square$

LEM 20 $\{E_i\hat{x}\}_{i=0}^{d}$ is an orthog basis for $M\hat{x}$.

pf By L19 and since

$$M\hat{x} = \sum_{i=0}^{d} E_iM\hat{x} \qquad \text{(ods)}$$

$\square$

DEF 21 For $0 \le i \le d$ define

$$\mathbb{1}_i^* = E_i \hat{x}$$

$$m_i = \| \mathbb{1}_i^* \|^2$$

Note

$$\hat{x} = \sum_{i=0}^{d} \mathbb{1}_i^*$$

$$m_i \neq 0 \qquad 0 \le i \le d$$

$$1 = \sum_{i=0}^{d} m_i$$

We have seen that both

$$\{\mathbb{1}_i\}_{i=0}^d, \qquad \{\mathbb{1}_i^*\}_{i=0}^d$$

are orthogonal bases for $M\hat{x}$. We now consider how these bases are related.

LEM 22 For $0 \le i, j \le d$

$$\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = f_i(\theta_j) m_j$$

pf

$$\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = \langle f_i(A)\hat{x}, \underset{\underset{E_j^2}{\|}}{E_j} \hat{x} \rangle$$

$$= \langle \underbrace{E_j f_i(A)}_{\underset{f_i(\theta_j) E_j}{\|}} \hat{x}, E_j \hat{x} \rangle$$

$$= f_i(\theta_j) \| E_j \hat{x} \|^2$$

$$= f_i(\theta_j) m_j. \qquad \square$$

We now give the transition matrices between our bases for $M\hat{x}$

LEM 23 For $0 \le j \le d$

(i) $$\mathbb{1}_j = \sum_{i=0}^{d} f_j(\theta_i)\, \mathbb{1}_i^*$$

(ii) $$\mathbb{1}_j^* = \sum_{i=0}^{d} \frac{f_i(\theta_j)\, m_j}{k_i}\, \mathbb{1}_i$$

pf (i)

$$\begin{aligned}
\mathbb{1}_j &= f_j(A)\,\hat{x} \\
&= I\, f_j(A)\,\hat{x} \\
&= \sum_{i=0}^{D} f_j(A)\, E_i \hat{x} \\
&= \sum_{i=0}^{d} f_j(A)\, E_i \hat{x} \\
&= \sum_{i=0}^{d} f_j(\theta_i)\, \underbrace{E_i \hat{x}}_{= \mathbb{1}_i^*}
\end{aligned}$$

$$I = \sum_{i=0}^{D} E_i$$

(ii) Use LEM 22. write

$$\mathbb{1}_j^* = \sum_{h=0}^{d} \alpha_h \mathbb{1}_h \qquad \alpha_h \in \mathbb{C}$$

For $0 \le i \le d$

$$\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = \left\langle \mathbb{1}_i, \sum_{h=0}^{d} \alpha_h \mathbb{1}_h \right\rangle$$

$$= \overline{\alpha_h} k_i$$

By LEM 22

$$\langle \mathbb{1}_i, \mathbb{1}_j^* \rangle = f_i(\theta_j) m_j$$

So

$$\overline{\alpha_i} = \frac{f_i(\theta_j) m_j}{k_i} = \alpha_i \qquad \square$$

The polynomials $\{f_i\}_{i=0}^d$ are "orthogonal" in the following sense.

LEM 24 For $0 \le r, s \le d$

(i) $$\sum_{i=0}^d f_r(\theta_i)\, f_s(\theta_i)\, m_i = \delta_{rs} k_r$$

(ii) $$\sum_{i=0}^d f_i(\theta_r)\, f_i(\theta_s)\, k_i^{-1} = \delta_{rs} m_r^{-1}$$

pf (i)

$$\delta_{rs} k_r = \langle \mathbb{1}_r, \mathbb{1}_s \rangle$$

$$= \left\langle \sum_{i=0}^d f_r(\theta_i)\, \mathbb{1}_i^*, \sum_{j=0}^d f_s(\theta_j)\, \mathbb{1}_j^* \right\rangle$$

$$= \sum_{i=0}^d f_r(\theta_i)\, f_s(\theta_i)\, m_i$$

(ii)

$$\delta_{rs} m_r = \langle \mathbb{1}_r^*, \mathbb{1}_s^* \rangle$$

$$= \left\langle \sum_{i=0}^{d} \frac{f_i(\theta_r) m_r}{k_i} \mathbb{1}_i, \sum_{j=0}^{d} \frac{f_j(\theta_s) m_s}{k_j} \mathbb{1}_j \right\rangle$$

$$= \sum_{i=0}^{d} \frac{f_i(\theta_r) f_i(\theta_s) m_r m_s}{k_i^2} k_i$$

Result follows. $\square$

DEF 25 Γ is said to be <u>distance-regular</u> whenever $\forall x \in X$

(i) Γ is distance-regular with respect to x;

(ii) The intersection numbers of Γ with respect to x do not depend on x.

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