# LECTURE 4 Wed Sept 11



We continue to study the graph $\Gamma = P_D$, a path of length $D$. $x =$ end-vertex of $\Gamma$.

Recall $q \in \mathbb{C}$ is a $(2D+4)$ - prim root of 1.

$$A^* = \mathrm{diag}(\theta_0^*, \ldots, \theta_D^*)$$

$$\theta_i^* = q^{i+1} + q^{-i-1} \qquad 0 \le i \le D$$

Thm 11 the eigenvalues of $\Gamma = P_D$ are

$$\theta_i = q^{i+1} + q^{-i-1} \qquad (0 \le i \le D)$$

For this ordering $A^*$ is a dual adjacency matrix wrt $x$.

Moreover

$$E_i A^* E_i = 0 \qquad (0 \le i \le D)$$

"dual bipartite"

Proof: For the time being let

$\{\theta_i\}_{i=0}^{\mathcal{D}}$ denote any ordering of the eigenvalues

of $\Gamma$. We saw earlier $\mathcal{D} \geq D$.

But $\mathcal{D}+1 \leq |X| = D+1$

So $$\mathcal{D} = D$$

and $$\dim E_iV = 1 \qquad 0 \leq i \leq D$$

Draw a diagram on the nodes $0, 1, \ldots, D$.

For $0 \leq i \leq D$ node $i$ represents $\theta_i$ or $E_i$

For $0 \leq i, j \leq D$ attach nodes $i, j$ by an arc

$i \frown j$ whenever

$$E_iA^*E_j \neq 0$$

(So $i$ gets a loop $i$ whenever $E_iA^*E_i \neq 0$)

Note that $$E_iA^*E_j = 0 \quad \text{iff} \quad E_jA^*E_i = 0$$

Since $$\overline{(E_iA^*E_j)}^{\,t} = E_jA^*E_i$$

therefore the diagram is undirected.

the diagram is connected by LEM 10

<u>claim 1</u> For $0 \le i, j \le D$ assume nodes $i, j$ are connected by an arc. Then

$$\theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 = -(q-q^{-1})^2$$

<u>pf cl</u> Consider eq (1) in Prop 9

$$0 = E_i \left( LHS - RHS \right) E_j$$

$$= \underbrace{E_i A^* E_j}_{\neq 0} \Big( \underbrace{\theta_i^2 - \beta \theta_i \theta_j + \theta_j^2 + (q-q^{-1})^2}_{\text{must be } 0} \Big)$$

<u>claim 2</u> Each node $i$ in diagram is connected by an arc to at most 2 nodes in the diagram.

<u>pf cl</u> For each node $j$ that is connected to node $i$ by an arc, $\theta_j$ is a root of the quadratic polynomial

$$\lambda^2 - \beta \theta_i \lambda + \theta_i^2 + (q-q^{-1})^2.$$

claim 3 In the diagram, assume node $i$ is adjacent nodes $r, s$ $(r \neq s)$

then

$$\theta_r - \beta\theta_i + \theta_s = 0$$

pf of cl Both

$$\theta_i^2 - \beta\theta_i\theta_r + \theta_r^2 = -(q-q^{-1})^2,$$

$$\theta_i^2 - \beta\theta_i\theta_s + \theta_s^2 = -(q-q^{-1})^2.$$

Take the difference:

$$\theta_r^2 - \theta_s^2 = \beta\theta_i(\theta_r - \theta_s)$$

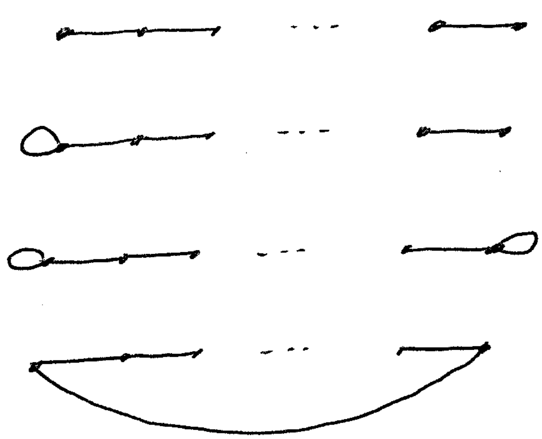
$$\parallel$$

$$(\theta_r - \theta_s)(\theta_r + \theta_s)$$

$\theta_r \neq \theta_s$ so

$$\theta_r + \theta_s = \beta\theta_i \quad \checkmark$$

So far, the possible diagrams are



(022)

In any case, WLOG our ordering $\{\theta_i\}_{i=0}^{D}$ satisfies

$$\theta_{i-1} \frown \theta_i \qquad 1 \le i \le D$$

By claim 1

$$\theta_{i-1}^2 - \beta \theta_{i-1}\theta_i + \theta_i^2 = -(q-q^{-1})^2 \qquad (1 \le i \le D) \qquad *$$

By claim 3

$$\theta_{i-1} - \beta \theta_i + \theta_{i+1} = 0 \qquad 1 \le i \le D-1 \qquad **$$

By (**) and since $\beta = q+q^{-1}$, $\exists\ a, b \in \mathbb{C}$ s.t.

$$\theta_i = a q^i + b q^{-i} \qquad (0 \le i \le D)$$

Evaluate (*) using this to get

$$ab = 1$$

So

$$\theta_i = aq^i + a^{-1}q^{-i} \qquad (0 \le i \le D)$$

All diagonal entries of $A$ are $0$. Therefore

$$0 = \text{trace}(A)$$

$$= \sum_{i=0}^{D} \theta_i$$

$$= a(1+q+q^2+\cdots+q^D) + \underbrace{a^{-1}(1+q^{-1}+q^{-2}+\cdots+q^{-D})}_{\shortparallel}$$

$$a^{-1}q^{-D}(1+q+q^2+\cdots+q^D)$$

$$= (a + a^{-1}q^{-D}) \frac{q^{D+1}-1}{q-1}$$

But

$$q^{D+1}-1 = -q^{-1}-1 \neq 0$$

So

$$a + a^{-1}q^{-D} = 0$$

So

$$a^2 = -q^{-D} = q^2$$

So $\quad a = \mp q$

For $\quad a = q$

$$\theta_i = q^{i+1} + q^{-i-1} \qquad (0 \le i \le D)$$

For $\quad a = -q \quad$ get same list in reverse order

So WLOG

$$\theta_i = q^{i+1} + q^{-i-1} \qquad (0 \le i \le D)$$

<u>claim 4</u> $\quad$ In the diagram,

no loop at $\theta_0$ or $\theta_D$. $\quad$ Also $\theta_0, \theta_D$ are not connected by an arc, provided that $D \ge 2$.

<u>pf cl</u> $\quad$ Use claim 3. $\quad$ One checks

$$\theta_0 - \beta\theta_0 + \theta_1 \ne 0$$

$$\theta_D - \beta\theta_D + \theta_{D-1} \ne 0$$

$$\theta_1 - \beta\theta_0 + \theta_D \ne 0$$

By the above claim the diagram is

$$\theta_0 - \theta_1 - \theta_2 \quad \cdots \quad \theta_{D-1} - \theta_D$$

So $\quad E_i A^* E_j = 0 \quad$ if $\quad |i-j| = 1 \quad (0 \le i,j \le D)$ $\quad \square$

Until further notice $\Gamma = (X, R)$ denotes any connected graph.

To avoid trivialites assume $|X| \geq 2$

Write
$$\mathbb{1} = \sum_{y \in X} \hat{y}$$
"all 1's vector"

Fix $x \in X$ and write $M^* = M^*(x)$, $T = T(x)$,
$d = D_x$

For $i \in \mathbb{Z}$ define
$$\Gamma_i(x) = \{ y \in X \mid \partial(x,y) = i \}$$

So
$$\Gamma_i(x) = \phi \text{ if } i < 0 \text{ or } i > d$$

Also
$$\Gamma_0(x) = \{x\} \qquad \Gamma_1(x) = \Gamma(x)$$

For $0 \leq i \leq d$
$\{ \hat{y} \}_{y \in \Gamma_i(x)}$ is a basis for $E_i^* V$

Define
$$k_i = k_i(x) = |\Gamma_i(x)|$$

So $k_i = \dim E_i^* V$

Note

$$k_0 = 1$$

$$k_1 = k(x) = \text{valency of } x$$

$$|X| = \sum_{i=0}^{d} k_i$$

DEF 12 $\Gamma$ is said to be <u>distance-regular with respect to $x$</u> whenever for $0 \le i \le d$ and $y \in \Gamma_i(x)$

$c_i := |\Gamma(y) \cap \Gamma_{i-1}(x)|$ is independent of $y$

$a_i := |\Gamma(y) \cap \Gamma_i(x)|$ ...

$b_i := |\Gamma(y) \cap \Gamma_{i+1}(x)|$ ...

Call

$a_i, b_i, c_i$ $\quad 0 \le i \le d$

the <u>intersection numbers</u> <u>of $\Gamma$ wrt $x$</u>