Lecture 3 Monday Sept 9

Recall our connected graph $\Gamma = (X, R)$

Fix $x \in X$, write $M^* = M^*(x)$, $T = T(x)$

Earlier we saw that the standard module V is an orthogonal dir sum of irred T-modules.

Let W denote an irred T-module.

We associate with W the following parameters:

name	meaning
endpt of W	$\min\{i \mid 0 \le i \le D_x, \ E_i^* W \neq 0\}$
diameter of W	$\lvert\{i \mid 0 \le i \le D_x, \ E_i^* W \neq 0\}\rvert - 1$

$\exists$ unique irred T-module with endpt 0

This T-module has diameter D_x.

Call this T-module <u>primary</u>

Fix an ordering $\{\theta_i\}_{i=0}^{D}$ of the eigenvalues of Γ.

Def 4. By a <u>dual adjacency matrix of Γ</u> with respect to x and $\{\theta_i\}_{i=0}^{D}$ we mean a matrix $A^* \in Mat_X(\mathbb{C})$ such that

(i) A^* generates M^*

(ii) $E_i A^* E_j = 0$ if $|i-j|>1$ $\quad (0 \le i,j \le D)$

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Until further notice A^* denotes a dual adj matrix for Γ wrt x and $\{\theta_i\}_{i=0}^{D}$.

By (i) alone

A^* is diagonal.

The eigenspaces of A^* are the subconstituents of Γ wrt x.

By (ii) alone,

$$A^* E_i V \subseteq E_{i-1}V + E_i V + E_{i+1} V \quad (0 \le i \le D) \qquad ☆☆$$

For $0 \le i \le D_x$ let θ_i^* denote the eigenvalue of A^* for the eigenspace $E_i^* V$. Then

$$A^* = \sum_{i=0}^{D_x} \theta_i^* E_i^*$$

By constr $\{\theta_i^*\}_{i=0}^{D_x}$ are mutually distinct.

The $\{E_i^*\}_{i=0}^{D_x}$ are the primitive idempotents of A^*

Obs

T is gen by A, A^*.

Ex (4-cycle) Γ has a dual adj matrix with

$$\theta_0^* = 2, \quad \theta_1^* = 0, \quad \theta_2^* = -2$$

Next goal: show that A, A^* act on each irred T-module as a TD pair.

Let W denote an irred T-module.

We associate with W the following parameters:

name	meaning
dual endpt of W	$\min\{i \mid 0 \le i \le D, \ E_i W \ne 0\}$
dual diameter of W	$\lvert\{i \mid 0 \le i \le D, \ E_i W \ne 0\}\rvert - 1$

[it turns out
diam of W = dual diam of W
but we won't assume this]

LEM 5 Let W denote an irred T-module with endpt r, dual endpt t, diam δ, dual diam d.

(i) For $0 \le i \le D$,

$$E_i^* W \neq 0 \quad \text{iff} \quad r \le i \le r+\delta$$

(ii) For $0 \le i \le D$

$$E_i W \neq 0 \quad \text{iff} \quad t \le i \le t+d$$

pf (i) By constr

$$E_i^* W = 0 \qquad 0 \le i \le r-1$$

$$E_r^* W \neq 0$$

Suppose $\exists\, i$ $(r+1 \le i \le r+\delta)$ such that $E_i^* W = 0$

Put $W' = E_r^* W + \cdots + E_{r+i-1}^* W$

Obs $AW' \subseteq W'$ and $A^* W' \subseteq W'$

so W' is T-module.

By constr $W' \neq 0$, $W' \neq W$ $W' \subseteq W$

contradiction.

So $\qquad E_i^* W \neq 0 \qquad r \le i \le r+\delta$

By def of δ
$$E_i^* W = 0 \qquad r+\delta < i \le D_x$$

(ii) Sim to pf of (i) $\square$

COR 6 Let W denote an irred T-module. Then A, A^* act on W as a TD pair.

pf Check A, A^* actions satisfy the axioms of a TD pair.

Let r, t, δ, d be as in Lem 5

- Each of A, A^* is diagonalizable on W, since it is diagonalizable on V

- Define $V_i = E_{t+i} W$ for $0 \le i \le d$. Then $\{V_i\}_{i=0}^{d}$ is an ordering of the eigenspaces for A on W.

By ☆☆

$$A^* V_i \subseteq V_{i-1} + V_i + V_{i+1} \qquad 0 \le i \le d$$

where $V_{-1} = 0$, $V_{d+1} = 0$

- Define $V_i^* = E^*_{r+i} W$ for $0 \le i \le \delta$. Then $\{V_i^*\}_{i=0}^{\delta}$ is an ordering of the eigenspaces of A^* on W

By the triangle ineq

$$A V_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \qquad 0 \le i \le \delta$$

where $V_{-1}^* = 0$, $V_{\delta+1}^* = 0$

- The irreducibility cond. for TD pairs is satisfied by W, since W is irred. as a T-module and T is generated by A, A^*. $\square$

Note For $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq 0$, $\alpha A^* + \beta I$ is a dual adj matrix for Γ wrt x and $\{\theta_i\}_{i=0}^{D}$.

DEF 7 the graph Γ is said to be <u>Q-polynomial with respect to x and $\{\theta_i\}_{i=0}^{D}$</u> whenever $\exists$ a dual adj matrix of Γ wrt x and $\{\theta_i\}_{i=0}^{D}$

Def 8 For an integer $D \geq 0$ define the graph

P_D :

P_D has $D+1$ vertices and diameter D

Next goal: show that $\Gamma = P_D$ is Q-poly with respect to each end-vertex. Let x = end vertex.

For $0 \leq i \leq D$ define

x_i = unique vertex of P_D at distance i from x

Relative the ordering $\{x_i\}_{i=0}^{D}$,

$$A = \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & 0 & \\ & 1 & 0 & 1 & & \\ & & 1 & \ddots & \ddots & \\ & 0 & & \ddots & \ddots & 1 \\ & & & & 1 & 0 \end{pmatrix}$$

$$E_i^* = \operatorname{diag}(0, \ldots, 0, \underset{\substack{\uparrow \\ \text{coord } i}}{1}, 0, \ldots 0) \qquad (0 \leq i \leq D)$$

Obs

$$E_i^* A E_j^* = 0 \quad \text{iff} \quad |i-j| = 1 \qquad (0 \le i, j \le D)$$

" Γ is bipartite "

Pick $q \in \mathbb{C}$ that is a $(2D+4)$-primitive root of 1. So

$$q^{2D+4} = 1, \qquad q^{D+2} = -1$$

For $0 \le i \le D$ define

$$\theta_i^* = q^{i+1} + q^{-i-1}$$
$$= q\left(q^i - q^{D-i}\right)$$

Obs

$$\theta_{D-i}^* = -\theta_i^* \qquad (0 \le i \le D)$$

and

$$\theta_i^* - \theta_j^* = (q^i - q^j)(q^{i+1} - q^{-j-1})\, q^{-i} \qquad (0 \le i, j \le D)$$

So

$\{\theta_i^*\}_{i=0}^{D}$ are mutually distinct.

Write

$$\beta = q + q^{-1}$$

One checks

$$\theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1} = 0 \qquad (1 \le i \le D-1)$$

$$\theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i = -(q-q^{-1})^2 \qquad (1 \le i \le D)$$

Define

$$A^* = \operatorname{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_D) = \sum_{i=0}^{D} \theta^*_i E^*_i$$

We will show that A^* is a dual adj matrix of Γ with respect to x.

Prop 9 Both

$$A^2A^* - \beta AA^*A + A^*A^2 = -(q-q^{-1})^2 A^* \qquad (1)$$

$$A^{*2}A - \beta A^*AA^* + AA^{*2} = -(q-q^{-1})^2 A \qquad (2)$$

pf Just multiply it out. Assume $D \geq 1$ else triv.

Here are some details.

check (2): write

$$\Delta = LHS - RHS$$

show $\Delta = 0$

View $\Delta = I \Delta I \qquad I = \sum_{i=0}^{D} E_i^*$

$$= \sum_{0 \leq i,j \leq D} E_i^* \Delta E_j^*$$

For $0 \leq i, j \leq D$ show

$$E_i^* \Delta E_j^* = 0$$

Using

$$E_i^* A^* = \theta_i^* E_i^* \qquad A^* E_j^* = \theta_j^* E_j^*$$

we obtain

$$E_i^* \Delta E_j^* =$$

$$\underbrace{E_i^* A E_j^*}_{\substack{\| \\ 0 \\ \text{if } |i-j| \neq 1}} \Big(\underbrace{\theta_i^{*2} - \beta \theta_i^* \theta_j^* + \theta_j^{*2} + (q-q^{-1})^2}_{\substack{\| \\ 0 \\ \text{if } |i-j| = 1}} \Big)$$

Check (1) Write $\Delta = LHS - RHS$

show $\Delta = 0$

For $0 \le i,j \le D$ show

$$E_i^* \Delta E_j^* = 0$$

note $E_i^* \Delta E_j^* = 0$ unless $i=j$ or $|i-j| = 2$

For $0 \le i \le D-2$

$$E_i^* \Delta E_{i+2}^* = E_i^* A^2 E_{i+2}^* \big(\underbrace{\theta_i^* - \beta \theta_{i+1}^* + \theta_{i+2}^*}_{=0} \big)$$

$$= 0$$

$$E_{i+2}^* \Delta E_i^* = E_{i+2}^* A^2 E_i^* \big(\theta_{i+2}^* - \beta \theta_{i+1}^* + \theta_i^* \big)$$

$$= 0$$

For $1 \le i \le D-1$

$$E_i^* \Delta E_i^* = E_i^* \Big(2\theta_i^* - \beta \big(\underbrace{\theta_{i-1}^* + \theta_{i+1}^*}_{= \beta \theta_i^*} \big) + 2\theta_i^* + (q-q^{-1})^2 \theta_i^* \Big)$$

$$= E_i^* \theta_i^* \big(\underbrace{4 - \beta^2 + (q-q^{-1})^2}_{=0} \big)$$

$\beta = q + q^{-1}$

$$= 0$$

9/9/13

13

$$E_0^* \Delta E_0^* = E_0^* \Big(\underbrace{\theta_0^* - \beta\theta_1^* + \theta_0^* + (q-q^{-1})^2\theta_0^*}_{\substack{\| \\ \theta_0^*(q^2+q^{-2}) - \theta_1^*(q+q^{-1})}} \Big)$$

$= 0$

$= 0$

$$E_0^* \Delta E_0^* = E_0^* \Big(\underbrace{\theta_D^* - \beta\theta_{D-1}^* + \theta_D^* + (q-q^{-1})^2\theta_D^*}_{\theta_D^*(q^2+q^{-2}) - \theta_{D-1}^*(q+q^{-1})} \Big)$$

$$\theta_D^* = -\theta_0^*, \qquad \theta_{D-1}^* = -\theta_1^*$$

$= 0$

$= 0$

$\square$

LEM 10 The T-module V is irreducible.

pf

$$V = \sum_W W \qquad \text{dir sum of irred } T\text{-modules}$$

$$\mathbb{F}\hat{x} = E_0^* V = \sum_W E_0^* W \qquad (ds)$$

So $\exists$ irred T-module W with endpt 0

$\hat{x} \in W$

Fa $0 \le i \le D$

$$\hat{x}_i = E_i^* A^i \hat{x} \quad \in W$$

So

$$W = \text{Span}\{\hat{x}_i\}_{i=0}^{D} = V \qquad \square$$