LEC 39 FRIDAY DEC 13

Recall $U_q = U_q(sl_2)$

U_q-module $V = V_{d,1}$

dual space V^* becomes U_q-module

For V and V^* we have 12 bases

Next goal: transition matrices

Note: Given any basis $\{u_i\}_{i=0}^d$ for V, dual basis $\{u_i^*\}_{i=0}^d$ for V^*

- - - - $\{v_i\}_{i=0}^d$ for V - - - $\{v_i^*\}_{i=0}^d$ - -

Let S = trans matrix $\{u_i\}_{i=0}^d$ to $\{v_i\}_{i=0}^d$

then S^t = - - $\{v_i^*\}_{i=0}^d$ to $\{u_i^*\}_{i=0}^d$

Note For any basis for V or V^*

the trans matrix from that basis to its

inversion is $Z = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$

Thm 91 In the table below we display some transition matrices between bases for V. Each trans matrix is diagonal. For $0 \le i \le d$ the (i,i)-entry is given:

trans mat	(i,i)-entry
$[y]_{row} \to [y]_{col}$	$(-1)^i q^{i(1-d)} \begin{bmatrix} d \\ i \end{bmatrix}_q^{-1} \dfrac{(\eta_z, \zeta_x^*)}{(\eta_z, \zeta_y^*)(\eta_y, \zeta_x^*)}$
$[y]_{col} \to [y]_{row}$	$(-1)^i q^{i(d-1)} \begin{bmatrix} d \\ i \end{bmatrix}_q \dfrac{(\eta_z, \zeta_y^*)(\eta_y, \zeta_x^*)}{(\eta_z, \zeta_x^*)}$
$[y]^{inv}_{row} \to [y]^{inv}_{col}$	$(-1)^{d-i} q^{(d-i)(1-d)} \begin{bmatrix} d \\ i \end{bmatrix}_q^{-1} \dfrac{(\eta_z, \zeta_x^*)}{(\eta_z, \zeta_y^*)(\eta_y, \zeta_x^*)}$
$[y]^{inv}_{col} \to [y]^{inv}_{row}$	$(-1)^{d-i} q^{(d-i)(d-1)} \begin{bmatrix} d \\ i \end{bmatrix}_q \dfrac{(\eta_z, \zeta_y^*)(\eta_y, \zeta_x^*)}{(\eta_z, \zeta_x^*)}$

(+ CP)

Pf $[y]_{row} \to [y]_{col}$:

Both bases are given in Prop 87.

Just compare.

$\square$

th 92 In the table below we display some trans matrices between bases for V^*. Each trans matrix is diagonal. For $0 \le i \le d$ the (i,i)-entry is given.

trans mat	(i,i)-entry
$[y]_{row} \to [y]_{col}$	$(-1)^i q^{i(d-1)} \begin{bmatrix} d \\ i \end{bmatrix}_q^{-1} \dfrac{(\eta_x, \eta_z^*)}{(\eta_x, \eta_y^*)(\eta_y, \eta_z^*)}$
$[y]_{col} \to [y]_{row}$	$(-1)^i q^{i(1-d)} \begin{bmatrix} d \\ i \end{bmatrix}_q \dfrac{(\eta_x, \eta_y^*)(\eta_y, \eta_z^*)}{(\eta_x, \eta_z^*)}$
$[y]_{row}^{inv} \to [y]_{col}^{inv}$	$(-1)^{d-i} q^{(d-i)(d-1)} \begin{bmatrix} d \\ i \end{bmatrix}_q^{-1} \dfrac{(\eta_x, \eta_z^*)}{(\eta_x, \eta_y^*)(\eta_y, \eta_z^*)}$
$[y]_{col}^{inv} \to [y]_{row}^{inv}$	$(-1)^{d-i} q^{(d-i)(1-d)} \begin{bmatrix} d \\ i \end{bmatrix}_q \dfrac{(\eta_x, \eta_y^*)(\eta_y, \eta_z^*)}{(\eta_x, \eta_z^*)}$
	$(+\ CP)$

pf Sim to th 91 $\square$

Th 93 In the table below we give some trans matrices between bases for V. Each trans matrix is lower triangular. For $0 \le j \le i \le d$ the (i,j)-entry is given

trans matrix	(i,j)-entry
$[y]_{row} \to [z]^{inv}_{row}$	$(-1)^j q^{j(1-i)} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{(\eta_z, \eta^*_x)}{(\eta_y, \eta^*_x)}$
$[y]_{col} \to [z]^{inv}_{col}$	$(-1)^{d-i} q^{(i-d)(d-j-1)} \begin{bmatrix} d-j \\ i-j \end{bmatrix}_q \dfrac{(\eta_x, \eta^*_y)}{(\eta_x, \eta^*_z)}$
$[y]^{inv}_{row} \to [x]_{row}$	$(-1)^j q^{j(i-1)} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{(\eta_x, \eta^*_z)}{(\eta_y, \eta^*_z)}$
$[y]^{inv}_{col} \to [x]_{col}$	$(-1)^{d-i} q^{(d-i)(d-j-1)} \begin{bmatrix} d-j \\ i-j \end{bmatrix}_q \dfrac{(\eta_z, \eta^*_y)}{(\eta_z, \eta^*_x)}$

(+ CP)

pf $[y]_{row} \to [z]_{row}^{inv}$:

$\{u_i\}_{i=0}^{d}$ $\{v_i\}_{i=0}^{d}$

By Prop 87

$$v_j = \frac{(-1)^j q^{-\binom{j}{2}}}{[j]_q^!} \frac{(\eta_z \eta_x^*)}{(\eta_y \eta_x^*)} A_x^j \eta_y \quad (0 \le j \le d)$$

Eval RHS :

Recall
$$\eta_y = \sum_{i=0}^{d} u_i$$

Also recall
$$A_x u_i = q^{-i}\left[i+1\right]_q u_{i+1} \qquad 0 \le i \le d-1,$$

$$A_x u_d = 0$$

Result follows

12/13/13
7

$[y]_{col} \to [z]^{inv}_{col}$:

Compute product of trans matrices

$$[y]_{col} \xrightarrow[\text{th } 92]{} [y]_{row} \xrightarrow[\text{earlier}]{} [z]^{inv}_{row} \xrightarrow[\text{th } 92]{} [z]^{inv}_{col}$$

$[y]^{inv}_{row} \to [x]_{row}$:

$\{u_i\}_{i=0}^{d}$ $\qquad$ $\{v_i\}_{i=0}^{d}$

By Prop 87

$$v_j = \frac{q^{\binom{j}{2}}}{[j]^!_q} \frac{(\eta_x, \eta_z^*)}{(\eta_y, \eta_z^*)} A_z^{*j} \eta_y \qquad (0 \le j \le d) \quad \star$$

Eval RHS :

Recall $\quad \eta_y = \sum_{i=0}^{d} u_i$

$$A_z u_i = -q^i [i+1]_q u_{i+1} \qquad 0 \le i \le d-1,$$

$$A_z u_d = 0$$

$[y]_{col}^{inv} \to [x]_{col}$:

Compute prod of trans matrices

$$[y]_{col}^{inv} \xrightarrow[\text{th } 92]{} [y]_{row}^{inv} \xrightarrow[\text{earlier}]{} [x]_{row} \xrightarrow[\text{th } 92]{} [x]_{col}$$

$\square$

th 94 In the table below we give some trans matrices between bases for V^*. Each trans matrix is lower triang. For $0 \le j \le i \le d$ the (i,j)-entry is given

trans mat	(i,j)-entry
$[y]_{row} \to [z]^{inv}_{row}$	$(-1)^j q^{j(i-1)} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{(\eta_x, \eta_z^*)}{(\eta_x, \eta_y^*)}$
$[y]_{col} \to [z]^{inv}_{col}$	$(-1)^{d-i} q^{(d-i)(d-j-1)} \begin{bmatrix} d-j \\ i-j \end{bmatrix}_q \dfrac{(\eta_y, \eta_x^*)}{(\eta_z, \eta_x^*)}$
$[y]^{inv}_{row} \to [x]_{row}$	$(-1)^j q^{j(1-i)} \begin{bmatrix} i \\ j \end{bmatrix}_q \dfrac{(\eta_z, \eta_x^*)}{(\eta_z, \eta_y^*)}$
$[y]^{inv}_{col} \to [x]_{col}$	$(-1)^{d-j} q^{(i-d)(d-j-1)} \begin{bmatrix} d-j \\ i-j \end{bmatrix}_q \dfrac{(\eta_y, \eta_z^*)}{(\eta_x, \eta_z^*)}$

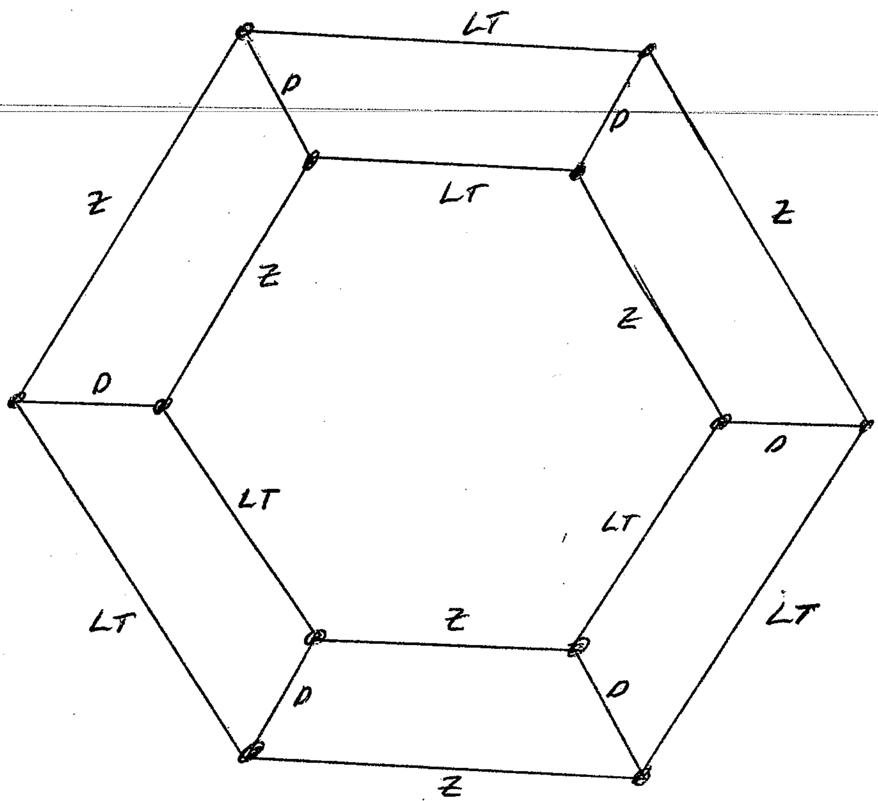
(+ CP)

pf Sim to th 93 $\square$

Given the above trans matrices,

we can get from any basis to any other

For V and $V^{\otimes 3}$



(each dot represents a basis)

For more info see

ter Finite, dual irred $U_q(sl_2)$ modules from the

equitable pt of view (Carson)

— ⊗ —