LEC 38 WED DEC 11

Def 79 Pick $\theta \in \{x, y, z\}$. Let

$[\theta]_{row}$ denote the unique basis for V from L 78

the basis $[\theta]_{row}$ for V^* is sim defined, with ζ_θ replaced

by ζ_θ^*

the inversion of $[\theta]_{row}$ is denoted $[\theta]_{row}^{inv}$.

LEM 80 In the table below we give 3 bases for V.

For each basis we give component 0 and d

basis for V	comp 0	comp d
$[x]_{row}$	$\zeta_y \dfrac{(\zeta_x, \zeta_z^*)}{(\zeta_y, \zeta_z^*)}$	$\zeta_z \dfrac{(\zeta_x, \zeta_y^*)}{(\zeta_z, \zeta_y^*)}$
$[y]_{row}$	$\zeta_z \dfrac{(\zeta_y, \zeta_x^*)}{(\zeta_z, \zeta_x^*)}$	$\zeta_x \dfrac{(\zeta_y, \zeta_z^*)}{(\zeta_x, \zeta_z^*)}$
$[z]_{row}$	$\zeta_x \dfrac{(\zeta_z, \zeta_y^*)}{(\zeta_x, \zeta_y^*)}$	$\zeta_y \dfrac{(\zeta_z, \zeta_x^*)}{(\zeta_y, \zeta_x^*)}$

pf Denote basis $[x]_{row}$ by $\{v_i\}_{i=0}^{d}$

So $v_i \in$ comp i of dec $[x]$

So $\exists\ \alpha, \beta \in \mathbb{F}$ s.t.

$$v_0 = \alpha \eta_y \qquad v_d = \beta \eta_z$$

Also

$$\eta_x = \sum_{i=0}^{d} v_i$$

Obs

$$(\eta_x, \eta_z^*) = \sum_{i=0}^{d} \underbrace{(v_i, \eta_z^*)}_{=0 \text{ if } i \neq 0}$$

$$= (v_0, \eta_z^*)$$

$$= \alpha(\eta_y, \eta_z^*)$$

So

$$\alpha = \frac{(\eta_x, \eta_z^*)}{(\eta_y, \eta_z^*)}$$

Similarly

$$\beta = \frac{(\eta_x, \eta_y^*)}{(\eta_z, \eta_y^*)}$$

Done for basis $[x]_{row}$. $\square$

LEM 81 In the table below we give 3 bases of V^*. For each basis we give components 0 and d.

basis of V^*	comp 0	comp d
$[x]_{row}$	$\zeta_y^* \dfrac{(\zeta_z, \zeta_x^*)}{(\zeta_z, \zeta_y^*)}$	$\zeta_z^* \dfrac{(\zeta_y, \zeta_x^*)}{(\zeta_y, \zeta_z^*)}$
$[y]_{row}$	$\zeta_z^* \dfrac{(\zeta_x, \zeta_y^*)}{(\zeta_x, \zeta_z^*)}$	$\zeta_x^* \dfrac{(\zeta_z, \zeta_y^*)}{(\zeta_z, \zeta_x^*)}$
$[z]_{row}$	$\zeta_x^* \dfrac{(\zeta_y, \zeta_z^*)}{(\zeta_y, \zeta_x^*)}$	$\zeta_y^* \dfrac{(\zeta_x, \zeta_z^*)}{(\zeta_x, \zeta_y^*)}$

pf Sim to L80 $\square$

DEF 82 For $\theta \in \{x, y, z\}$ let

$[\theta]_{col}$ denote the basis for V (resp V^*) that is dual to the basis $[\theta]_{row}^{inv}$ for V^* (resp V)

The inversion of $[\theta]_{col}$ is denoted $[\theta]_{col}^{inv}$.

LEM 83 In the table below we give 3 bases of V. For each basis we describe comp 0 and d.

basis of V	comp 0	comp d
$[x]_{col}$	$\dfrac{\eta_y}{(\eta_y, \eta_x^*)}$	$\dfrac{\eta_z}{(\eta_z, \eta_x^*)}$
$[y]_{col}$	$\dfrac{\eta_z}{(\eta_z, \eta_y^*)}$	$\dfrac{\eta_x}{(\eta_x, \eta_y^*)}$
$[z]_{col}$	$\dfrac{\eta_x}{(\eta_x, \eta_z^*)}$	$\dfrac{\eta_y}{(\eta_y, \eta_z^*)}$

pf $[x]_{col}$: By constr

comp 0 of $[x]_{col}$ = sc mult of η_y

.. d ... .. η_z

To find the scalars use Def 82 and L 81. $\square$

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LEM 84 In the table below we give 3 bases for V^*. For each basis we desc comp 0 and d.

basis for V^*	comp 0	comp d
$[x]_{col}$	$\dfrac{\eta_y^*}{(\eta_x, \eta_y^*)}$	$\dfrac{\eta_z^*}{(\eta_x, \eta_z^*)}$
$[y]_{col}$	$\dfrac{\eta_z^*}{(\eta_y, \eta_z^*)}$	$\dfrac{\eta_x^*}{(\eta_y, \eta_x^*)}$
$[z]_{col}$	$\dfrac{\eta_x^*}{(\eta_z, \eta_x^*)}$	$\dfrac{\eta_y^*}{(\eta_z, \eta_y^*)}$

pf Sim to L 83

$\square$

In summary —

LEM 85 Pick $\theta \in \{x, y, z\}$.
In the table below, we give a basis for V and basis for V^*. These bases are dual.

basis for V	basis for V^*
$[\theta]_{row}$	$[\theta]^{inv}_{col}$
$[\theta]_{col}$	$[\theta]^{inv}_{row}$
$[\theta]^{inv}_{row}$	$[\theta]_{col}$
$[\theta]^{inv}_{col}$	$[\theta]_{row}$

pf By def 82 and meaning of inv. $\square$

Prop 86 We have

$$\frac{(\eta_x, \eta_y^*)(\eta_y, \eta_z^*)(\eta_z, \eta_x^*)}{(\eta_x, \eta_z^*)(\eta_y, \eta_x^*)(\eta_z, \eta_y^*)} = (-1)^d q^{d(d+1)}$$

pf let $\{u_i\}_{i=0}^d$ denote basis $[y]_{row}$ for V

.. $\{v_i\}_{i=0}^d$ ... $[y]_{row}$ for V^*

These bases sat C79

By L80

$$u_0 = \eta_z \frac{(\eta_y, \eta_x^*)}{(\eta_z, \eta_x^*)}, \qquad u_d = \eta_x \frac{(\eta_y, \eta_z^*)}{(\eta_x, \eta_z^*)}$$

By L81

$$v_0 = \eta_z^* \frac{(\eta_x, \eta_y^*)}{(\eta_x, \eta_z^*)} \qquad v_d = \eta_x^* \frac{(\eta_z, \eta_y^*)}{(\eta_z, \eta_x^*)}$$

Eval COR 75 using this.

$\square$

Note By Prop 86 the scalars

$$(\eta_u, \eta_v^*) \qquad u, v \in \{x, y, z\} \qquad u \neq v$$

are det by

$$(\eta_x, \eta_y^*), \quad (\eta_y, \eta_z^*), \quad (\eta_z, \eta_x^*), \quad (\eta_y, \eta_x^*), \quad (\eta_z, \eta_y^*) \qquad (\star)$$

the scalars $(\star)$ are free in the following sense.

Given a sequence S of five nonzero scalars in $\mathbb{F}$ $\exists$ vectors

η_x, η_y, η_z and $\eta_x^*, \eta_y^*, \eta_z^*$ as in Def 76 such that

the sequence $(\star)$ is equal to S. (ex).

Prop 87. In the table below we give some bases for V. For $0 \le i \le d$ we desc comp i

basis for V	comp i (ver I)	comp i (ver II)
$[y]_{row}$	$\dfrac{q^{\binom{i}{2}}\,(\eta_y, \eta_x^*)}{[i]_q^!\,(\eta_z, \eta_x^*)}\, n_x^i \zeta_z$	$\dfrac{(-1)^{d-i}\, q^{-\binom{d-i}{2}}\,(\eta_y, \eta_z^*)}{[d-i]_q^!\,(\eta_x, \eta_z^*)}\, n_z^{d-i} \zeta_x$
$[y]_{col}$	$\dfrac{(-1)^i [d-i]_q^!\, q^{i(1-d)+\binom{i}{2}}}{[d]_q^!\,(\eta_z, \eta_y^*)}\, n_x^i \zeta_z$	$\dfrac{[i]_q^!\, q^{(d-i)(d-1)-\binom{d-i}{2}}}{[d]_q^!\,(\eta_x, \eta_y^*)}\, n_z^{d-i} \zeta_x$
$[y]_{row}^{inv}$	$\dfrac{(-1)^i q^{-\binom{i}{2}}\,(\eta_y, \eta_z^*)}{[i]_q^!\,(\eta_x, \eta_z^*)}\, n_z^i \zeta_x$	$\dfrac{q^{\binom{d-i}{2}}\,(\eta_y, \eta_x^*)}{[d-i]_q^!\,(\eta_z, \eta_x^*)}\, n_x^{d-i} \zeta_z$
$[y]_{col}^{inv}$	$\dfrac{[d-i]_q^!\, q^{i(d-1)-\binom{i}{2}}}{[d]_q^!\,(\eta_x, \eta_y^*)}\, n_z^i \zeta_x$	$\dfrac{(-1)^{d-i}[i]_q^!\, q^{(d-i)(1-d)+\binom{d-i}{2}}}{[d]_q^!\,(\eta_z, \eta_y^*)}\, n_x^{d-i} \zeta_z$

(+ CP)

pf $[y]_{row}$: Call this basis $\{v_i\}_{i=0}^d$.

Earlier we gave the actions of n_x, n_z on $\{v_i\}_{i=0}^d$

We have

$$v_i = \frac{q^{i-1}}{[i]_q} n_x v_{i-1} \qquad 1 \le i \le d$$

$$v_i = -\frac{q^{i-d+1}}{[d-i]_q} n_z v_{i+1} \qquad 0 \le i \le d-1$$

So $\therefore$

$$v_i = \frac{q^{\binom{i}{2}}}{[i]_q^!} n_x^i v_0$$

$$v_i = \frac{(-1)^{d-i} q^{-\binom{d-i}{2}}}{[d-i]_q^!} n_z^{d-i} v_d$$

By L80

$$v_0 = \frac{(\mathfrak{z}_y, \mathfrak{z}_x^*)}{(\mathfrak{z}_z, \mathfrak{z}_x^*)} \mathfrak{z}_z, \qquad v_d = \frac{(\mathfrak{z}_y, \mathfrak{z}_z^*)}{(\mathfrak{z}_x, \mathfrak{z}_z^*)} \mathfrak{z}_x$$

Result follows.

other bases are sim.

$\square$

Prop 88 In the table below, in each row we give a basis for V^*. For $0 \le i \le d$ we give comp i.

basis for V^*	comp i 1)	comp i (ver 2)
$[y]_{row}$	$\dfrac{q^{-\binom{i}{2}}(\eta_x,\eta_y^*)}{[i]_q^!\,(\eta_x,\eta_z^*)}\, n_x^i \eta_z^*$	$\dfrac{(-1)^{d-i} q^{\binom{d-i}{2}}(\eta_z,\eta_y^*)}{[d-i]_q^!\,(\eta_z,\eta_x^*)}\, n_z^{d-i} \eta_x^*$
$[y]_{col}$	$\dfrac{(-1)^i [d-i]_q^!\, q^{i(d-1)-\binom{i}{2}}}{[d]_q^!\,(\eta_y,\eta_z^*)}\, n_x^i \eta_z^*$	$\dfrac{[i]_q^!\, q^{(d-i)(1-d)+\binom{d-i}{2}}}{[d]_q^!\,(\eta_y,\eta_x^*)}\, n_z^{d-i} \eta_x^*$
$[y]_{row}^{inv}$	$\dfrac{(-1)^i q^{\binom{i}{2}}(\eta_z,\eta_y^*)}{[i]_q^!\,(\eta_z,\eta_x^*)}\, n_z^i \eta_x^*$	$\dfrac{q^{-\binom{d-i}{2}}(\eta_x,\eta_y^*)}{[d-i]_q^!\,(\eta_x,\eta_z^*)}\, n_x^{d-i} \eta_z^*$
$[y]_{col}^{inv}$	$\dfrac{[d-i]_q^!\, q^{i(1-d)+\binom{i}{2}}}{[d]_q^!\,(\eta_y,\eta_x^*)}\, n_z^i \eta_x^*$	$\dfrac{(-1)^{d-i}[i]_q^!\, q^{(d-i)(d-1)-\binom{d-i}{2}}}{[d]_q^!\,(\eta_y,\eta_z^*)}\, n_x^{d-i} \eta_z^*$

(+ CP)

pf Sim to Prop 87

□

Special Cases

Cor 89 We have

$$\Lambda_x^d \gamma_y = [d]_q^! \, q^{-\binom{d}{2}} \frac{(\gamma_y, \gamma_z^*)}{(\gamma_x, \gamma_z^*)} \gamma_x$$

$$\Lambda_z^d \gamma_y = (-1)^d [d]_q^! \, q^{\binom{d}{2}} \frac{(\gamma_y, \gamma_x^*)}{(\gamma_z, \gamma_x^*)} \gamma_z$$

(+ CP)

Cor 90 We have

$$n_x^d \eta_y^* = [d]_q^! \, q^{\binom{d}{2}} \frac{(\eta_z, \eta_y^*)}{(\eta_z, \eta_x^*)} \eta_x^*$$

$$n_z^d \eta_y^* = (-1)^d [d]_q^! \, q^{-\binom{d}{2}} \frac{(\eta_x, \eta_y^*)}{(\eta_x, \eta_z^*)} \eta_z^*$$

(+ CP)