Lecture 37 Monday Dec 9 12/9/13

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Recall $U_q = U_q(sl_2)$

U_q-module $V = V_{d,1}$

Dual space V^* becomes a $U_{q^{-1}}$-module

For V or V^* we have 12 bases

$[\theta]_{row}$, $[\theta]_{col}$, $[\theta]^{inv}_{row}$, $[\theta]^{inv}_{col}$

$\theta \in \{x, y, z\}$

We now give the actions of x, y, z wrt these bases.

Matrices in $Mat_{d+1}(\mathbb{F})$:

K_q is diagonal with (i,i)-entry q^{d-2i} $0 \le i \le d$.

Z has (i,j)-entry $\delta_{i+j,d}$ for $0 \le i,j \le d$

So $Z^2 = I$

For $d=3$

$$Z = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & 0 \\ 1 & & & \end{pmatrix}$$

For $B \in \mathrm{Mat}_{d+1}(\mathbb{F})$ we will discuss ZBZ

LEM 65 For $0 \le i, j \le d$ the following coincide

(i) the (i,j)-entry of ZBZ

(ii) the $(d-i, d-j)$-entry of B $\square$

pf ex

DEF 66 E_q is upper bidiag matrix in $\mathrm{Mat}_{d+1}(\mathbb{F})$
with (i,i)-entry q^{2i-d} for $0 \le i \le d$
and $(i-1,i)$-entry $q^d - q^{2i-d-2}$ for $1 \le i \le d$

ex for $d=3$

$$E_q = \begin{pmatrix} q^{-3} & q^3 - q^{-3} & 0 & 0 \\ 0 & q^{-1} & q^3 - q^{-1} & 0 \\ 0 & 0 & q & q^3 - q \\ 0 & 0 & 0 & q^3 \end{pmatrix}$$

We consider:

$$E_q, \quad E_{q^{-1}}, \quad E_q^t, \quad E_{q^{-1}}^t \qquad ☆$$

$$ZE_qZ, \quad ZE_{q^{-1}}Z, \quad ZE_q^tZ, \quad ZE_{q^{-1}}^tZ$$

LEM 67 The entries of above matrices are given in table below

matrix	entries $(i,i-1)$	(i,i)	$(i-1,i)$
E_q	0	q^{2i-d}	$q^d - q^{2i-2-d}$
$E_{q^{-1}}$	0	q^{d-2i}	$q^{-d} - q^{d-2i+2}$
E_q^t	$q^d - q^{2i-2-d}$	q^{2i-d}	0
$E_{q^{-1}}^t$	$q^{-d} - q^{d-2i+2}$	q^{d-2i}	0
ZE_qZ	$q^d - q^{d-2i}$	q^{d-2i}	0
$ZE_{q^{-1}}Z$	$q^{-d} - q^{2i-d}$	q^{2i-d}	0
ZE_q^tZ	0	q^{d-2i}	$q^d - q^{d-2i}$
$ZE_{q^{-1}}^tZ$	0	q^{2i-d}	$q^{-d} - q^{2i-d}$

pf ex $\square$

LEM 68. The matrices ✯ are described as follows

	upper/Lower bidiag	diag part	row/col sum
E_q	UB	K_q^{-1}	RS q^d
$E_{q^{-1}}$	UB	K_q	RS q^{-d}
E_q^t	LB	K_q^{-1}	CS q^d
$E_{q^{-1}}^t$	LB	K_q	CS q^{-d}
$Z E_q Z$	LB	K_q	RS q^d
$Z E_{q^{-1}} Z$	LB	K_q^{-1}	RS q^{-d}
$Z E_q^t Z$	UB	K_q	CS q^d
$Z E_{q^{-1}}^t Z$	UB	$K_{q^{-1}}$	CS q^{-d}

pf ex

□

thm 69 Consider x, y, z in U_q.

For the table below, in each row we give a basis for V along with the matrices that represent x, y, z rel the basis.

basis	x	y	z
$[y]_{row}$	E_q	K_q	$Z E_{q^{-1}} Z$
$[y]_{col}$	$Z E_{q^{-1}}^t Z$	K_q	E_q^t
$[y]_{row}^{inv}$	$Z E_q Z$	$K_{q^{-1}}$	$E_{q^{-1}}$
$[y]_{col}^{inv}$	$E_{q^{-1}}^t$	$K_{q^{-1}}$	$Z E_q^t Z$

(+ CP)

pf $[y]_{row}$: use L59

$[y]_{row}^{inv}$: conjugate by Z

$[y]_{col}$: Replace $q \to q^{-1}$ to get matrices that rep x, y, z rel basis $[y]_{row}^{inv}$ for V^*. For these matrices take transpose and use L64.

$[y]_{col}^{inv}$: conjugate by Z □

Note: Ref to Th 69, to get the corresp result for V^*, replace $q \to q^{-1}$ and leave the rest alone.

Next goal: Describe the actions of n_x, n_y, n_z on our 12 bases for V and V^*.

Def 70 Define $N_q \in \mathrm{Mat}_{d+1}(\mathbb{F})$ with $(i, i-1)$-entry $q^{1-i}[i]_q$, $1 \le i \le d$ and all other entries 0

ex For $d=3$

$$N_q = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & q^{-1}[2]_q & 0 & \\ & & q^{-2}[3]_q & 0 \end{pmatrix}$$

We consider

$$N_q \qquad N_{q^{-1}} \qquad N_q^{\,t} \qquad N_{q^{-1}}^{\,t}$$

$$ZN_qZ \qquad ZN_{q^{-1}}Z \qquad ZN_q^{\,t}Z \qquad ZN_{q^{-1}}^{\,t}Z$$

LEM 71 The entries of above matrices are given in the table below.

	$(i, i-1)$-entry	$(i-1, i)$-entry
N_q	$q^{1-i}[i]_q$	0
$N_{q^{-1}}$	$q^{i-1}[i]_q$	0
N_q^t	0	$q^{1-i}[i]_q$
$N_{q^{-1}}^t$	0	$q^{i-1}[i]_q$
ZN_qZ	0	$q^{i-d}[d-i+1]_q$
$ZN_{q^{-1}}Z$	0	$q^{d-i}[d-i+1]_q$
ZN_q^tZ	$q^{i-d}[d-i+1]_q$	0
$ZN_{q^{-1}}^tZ$	$q^{d-i}[d-i+1]_q$	0

pf ex

$\square$

Def 72 Define tridiag. $T_q \in \mathrm{Mat}_{d+1}(\mathbb{F})$ as follows.

$(i, i-1)$-entry $\quad q^{3i-2d-1}[i]_q \quad 1 \le i \le d$

$(i-1, i)$-entry $\quad -q^{3i-d-2}[d-i+1]_q \quad$ "

(i,i)-entry $\quad q^{2i-d}[i]_q[d-i+1]_q(q-q^{-1}) - q^{2i-d+1}[2i-d]_q \quad 0 \le i \le d$

Obs T_q has const row sum 0

One checks $\quad Z T_q Z = -T_{q^{-1}}$

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thm 73 Consider elements n_x, n_y, n_z of U_q.

For the table below, in each row we give a basis for V, along with the matrices that rep n_x, n_y, n_z rel the basis.

basis	n_x	n_y	n_z
$[y]_{row}$	N_q	T_q	$-Z N_{q^{-1}} Z$
$[y]_{col}$	$-Z N_{q^{-1}}^t Z$	T_q^t	N_q^t
$[y]^{inv}_{row}$	$Z N_q Z$	$-T_{q^{-1}}$	$-N_{q^{-1}}$
$[y]^{inv}_{col}$	$-N_{q^{-1}}^t$	$-T_{q^{-1}}^t$	$Z N_q^t Z$

(+ CP)

pf Sim to th 69. $\square$

Note Ref to th 73, to get corresp result for V^*, replace $q \to q^{-1}$ and leave rest alone.

Next goal: Consider action of bil form $(,)$ on our 12 bases for V and V^*.

Notation: For $n \geq i \geq 0$ define

$$\begin{bmatrix} n \\ i \end{bmatrix}_q = \frac{[n]_q^!}{[i]_q^! \, [n-i]_q^!}$$

LEM 74 Pick $\theta \in \{x, y, z\}$.
Let $\{u_i\}_{i=0}^d$ denote a $[\theta]_{row}$ basis for V
Let $\{v_i\}_{i=0}^d$ - - - - - V^*

Then

$$(u_r, v_s) = \delta_{r+s,d} (-1)^r q^{r(d-1)} \begin{bmatrix} d \\ r \end{bmatrix} (u_0, v_d)$$

$$0 \leq r, s \leq d$$

pf WLOG $\theta = y$

Case $r+s \neq d$: $(u_r, v_s) = 0$ by constr.

Case $r+s = d$: Recall $n_z^{\dagger} = -n_z$ so

For $1 \le i \le d$

$$\Big(\underbrace{n_z u_i}_{\parallel \text{ Lm } 73}, v_{d-i+1} \Big) = - \Big(u_i, \underbrace{n_z v_{d-i+1}}_{\parallel \text{ Lm } 73} \Big)$$

$$q^{d-i}[d-i+1]_q u_{i-1} \qquad\qquad q^{1-i}[i]_q v_{d-i}$$

Solving this recursion we get

$$(u_r, v_{d-r}) = (-1)^r q^{r(d-1)} \begin{bmatrix} d \\ r \end{bmatrix}_q (u_0, v_d)$$

Result follows. $\square$

COR 75 Ref to L 74

$$(u_d, v_0) = (-1)^d q^{d(d-1)} (u_0, v_d)$$

pf set $r=d$, $s=0$ in L 74 $\square$

Recall our 12 bases for V and V^*.

We now normalize these bases in more detail.

Def 76 For $\theta \in \{x, y, z\}$ let

η_θ denote a nonzero vector in $\Lambda_\theta^d V$

η_θ^* ... $\Lambda_\theta^d V^*$

LEM 77

(i) For dist $u, v \in \{x, y, z\}$

$$(\eta_u, \eta_v^*) \neq 0$$

(ii) Assume $d \geq 1$. Then for $\theta \in \{x, y, z\}$

$$(\eta_\theta, \eta_\theta^*) = 0$$

pf ex. $\square$

LEM 78 Pick $\theta \in \{x, y, z\}$.

$\exists$ unique basis $\{v_i\}_{i=0}^{d}$ of V such that

(i) For $0 \le i \le d$, $v_i \in$ component i of decomp $[\theta]$ of V

(ii) $\eta_\theta = \sum_{i=0}^{d} v_i$

pf $\exists$: Let $\{u_i\}_{i=0}^{d}$ denote a $[\theta]$row basis for V

Then $\sum_{i=0}^{d} u_i \in \eta_\theta^{d} V$

So $\exists\, k \in \mathbb{F}$ s.t.

$$\sum_{i=0}^{d} u_i = k\, \eta_\theta$$

By constr $k \neq 0$

Define $v_i = \frac{u_i}{k}$ $0 \le i \le d$.

$\{v_i\}_{i=0}^{d}$ is desired basis.

$\square$

Uniqueness: clear