LECTURE 36 FRIDAY DEC 6

Recall $U_q = U_q(sl_2)$

U_q-module $V = V_{d,1}$

dual space V^* is $U_{q^{-1}}$-module

For $\theta \in \{x, y, z\}$ recall

flag $\{\Lambda_\theta^{d-i} V\}_{i=0}^d$ on V

flag $\{\Lambda_\theta^{d-i} V^*\}_{i=0}^d$ on V^*

These flags are related as follows.

LEM 52 Pick $\theta \in \{x, y, z\}$. For $0 \le i \le d+1$

the following are orthog complements wrt $(,)$:

$\Lambda_\theta^i V, \qquad \Lambda_\theta^{d-i+1} V^*.$

pf WLOG $\theta = x$

By L37

Decomp $[y]$ of V is dual to decomp $[y]^{inv}$ of V^*

By L45

$\Lambda_x^i V$ = sum of components $i, i+1, \ldots, d$ of dec $[y]$ of V

$\Lambda_x^j V^*$ = - - - - $j, j+1, \ldots, d$ - - $[y]$ of V^*

= - - - $0, 1, \ldots, d-j$ - - $[y]^{inv}$ of V^*

Result follows. $\square$

Next goal:

Consider the subspaces

$$n_\theta^d V, \qquad n_\theta^d V^* \qquad \theta \in \{x, y, z\}$$

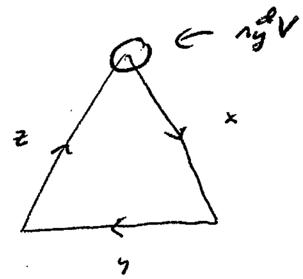
LEM 53

$n_y^d V$ is the eigenspace for z with eigval q^{-d}

... x ... q^{d}

(+ CP)

pf obs



$\square$

LEM 54

$n_y^d V^*$ is the eigenspace for z with eigval q^{d}

... x ... q^{-d}

(+ CP)

pf Apply L53 to U_q-module V^*. $\square$

LEM 55

(i) $n_y^d V$ is unique common eigenspace for x, z on V

(ii) $n_y^d V^*$ - - - - V^*

(+CP)

pf (i) let W = common eigspace for x, z on V.

x, z mult free on $V \Rightarrow \dim W = 1$

Also W is inv under x, z

Done by L51. $\square$

LEM 56

(i) $n_y^d V$ is the kernel of n_y on V

(ii) $n_y^d V^*$ - - - V^*

(+ CP)

pf
(i) Let $\{V_i\}_{i=0}^d$ denote decomp $[z]$ of V.

Recall $n_y V_i = V_{i+1}$ $0 \le i \le d-1$, $n_y V_d = 0$

$\ker$ of $n_y = V_d = n_y^d V$

(ii) Sim. $\square$

Next goal: For V and V^* we define 12 bases, denoted

$$[\theta]_{row}, \quad [\theta]_{col}, \quad [\theta]_{row}^{inv}, \quad [\theta]_{col}^{inv}$$

$$\theta \in \{x, y, z\}$$

DEF 57 Pick $\theta \in \{x, y, z\}$. A basis $\{v_i\}_{i=0}^d$ for V is called $[\theta]_{row}$ whenever:

(i) For $0 \le i \le d$, v_i is contained in component i of the decomp $[\theta]$

(ii) $\sum_{i=0}^{d} v_i \in \eta_\theta^d V$

A basis $[\theta]_{row}$ for V^* is similarly defined.

By a $[\theta]_{row}^{inv}$ basis we mean the inversion of a $[\theta]_{row}$ basis.

— o —

need to show bases in Def 57 exist

LEM 58 Consider decomp $[y]$ of V.

For $0 \le i \le d$ let v_i = any vector in ith component. TFAE

(i) $\sum_{i=0}^{d} v_i \in \Lambda_y^d V$

(ii) $(z - q^{2i-d}) v_i = (q^{-d} - q^{2i+2-d}) v_{i+1}$ for $0 \le i \le d-1$

(iii) $n_x v_i = q^{-i} [i+1]_q v_{i+1}$ for $0 \le i \le d-1$

(iv) $(x - q^{2i-d}) v_i = (q^d - q^{2i-2-d}) v_{i-1}$ for $1 \le i \le d$

(v) $n_z v_i = -q^{d-i} [d-i+1]_q v_{i-1}$ for $1 \le i \le d$

Now assume (i)-(v) hold. Then $\{v_i\}_{i=0}^{d}$ are all 0 or all nonzero.

pf By constr

$$y v_i = q^{d-2i} v_i \qquad 0 \le i \le d$$

Also $x v_0 = q^{-d} v_0$, $\quad z v_d = q^d v_d$

Abbr

$$\eta = \sum_{i=0}^{d} v_i$$

(i) $\Leftrightarrow$ (ii) By L 53

$$z \in n_y^d V \quad \text{iff} \quad Zz = q^{-d} z$$

Obs

$$(Z - q^{-d}) z = \sum_{i=0}^{d-1} w_i$$

where

$$w_i = \left(Z - q^{2i-d}\right) v_i + \left(q^{2i+2-d} - q^{-d}\right) v_{i+1} \qquad 0 \le i \le d-1$$

Obs

$$w_i \in \text{comp } i+1 \text{ of } [y]$$

So

$$(Z - q^{-d}) z = 0 \quad \Leftrightarrow \quad w_i = 0 \quad (0 \le i \le d-1)$$

Result follows.

(ii) $\Leftrightarrow$ (iii) Obs

$$Z = y^{-1} - q(q - q^{-1}) n_x y^{-1}$$

So for $0 \le i \le d-1$

$$(Z - q^{2i-d}) v_i = -q(q-q^{-1}) q^{2i-d+1} n_x v_i$$

(i) $\Leftrightarrow$ (iv) Sim to pf of (i) $\Leftrightarrow$ (ii)

(iv) $\Leftrightarrow$ (v) Sim to pf of (ii) $\Leftrightarrow$ (iii)

Last assertion routine $\square$

LEM 59 Let $\{v_i\}_{i=0}^{d}$ denote vectors in V, not all 0.

TFAE

(i) $\{v_i\}_{i=0}^{d}$ is a $[y]_{row}$ basis for V

(ii) $yv_0 = q^d v_0$ and $(z - q^{2i-d})v_i = (q^{-d} - q^{2i+2-d})v_{i+1}$ for $0 \le i \le d-1$

(iii) $yv_0 = q^d v_0$ and $n_x v_i = q^{-i}[i+1]_q v_{i+1}$ for $0 \le i \le d-1$

(iv) $yv_d = q^{-d} v_d$ and $(x - q^{2i-d})v_i = (q^d - q^{2i-2-d})v_{i-1}$ for $1 \le i \le d$

(v) $yv_d = q^{-d} v_d$ and $n_z v_i = -q^{d-i}[d-i+1]_q v_{i-1}$ for $1 \le i \le d$

Now assume (i) – (v) hold. Then

$$z v_d = q^d v_d, \quad n_x v_d = 0, \quad x v_0 = q^{-d} v_0, \quad n_z v_0 = 0$$

pf Each cond (i)–(v) implies that for $0 \le i \le d$ v_i is contained in component i of $[y]$. Now these conds are equiv by L 58.

Last assertion is routine. $\square$

LEM 60 For $\theta \in \{x, y, z\}$, $\exists$ $[\theta]_{row}$ basis for V and V^*

Pf WLOG underlying v.s. is V, and $\theta = y$.

Pick $0 \neq v_0 \in$ comp 0 of dec $[y]$ for V.

For $0 \leq i \leq d-1$ define v_{i+1} by

$$n_x v_i = q^{-i} [i+1]_q v_{i+1}$$

Then $\{v_i\}_{i=0}^d$ satisfies L59 (iii).

Now $\{v_i\}_{i=0}^d$ is a $[y]_{row}$ basis for V by L59 (i). $\square$

— o —

Let $\{u_i\}_{i=0}^d$ denote a basis for V

Let $\{v_i\}_{i=0}^d$ ... V^*

These bases are <u>dual</u> rel $(,)$ whenever

$$(u_i, v_j) = \delta_{ij} \quad \text{for } 0 \leq i, j \leq d$$

DEF 61 For $\theta \in \{x, y, z\}$.

A basis for V is called $[\theta]_{col}$ whenever its dual is a $[\theta]^{inv}_{row}$ basis for V^*

... V^* ... --

By a $[\theta]^{inv}_{col}$ basis we mean the inversion of a $[\theta]_{col}$ basis V.

Consider uniqueness of our bases

LEM 62 Let $\{v_i\}_{i=0}^d$ denote a $[y]_{row}$ basis for V.

Let $\{v_i'\}_{i=0}^d$ denote any vectors in V. TFAE

(i) the sequence $\{v_i'\}_{i=0}^d$ is a $[y]_{row}$ basis for V

(ii) $\exists\ 0 \neq \alpha \in \mathbb{F}$ s.t $v_i' = \alpha v_i$ for $0 \le i \le d$

pf Use L59. $\square$

Result sim to L62 applies to the bases

$[y]_{row}^{inv}$, $[y]_{col}$, $[y]_{col}^{inv}$ (+ CP)

COR 63 Pick $\theta \in \{x, y, z\}$ In the table below,

for each row we give a basis for V and its dual basis for V^*

basis for V	dual basis for V^*
$[\theta]_{row}$	$[\theta]_{col}^{inv}$
$[\theta]_{col}$	$[\theta]_{row}^{inv}$
$[\theta]_{row}^{inv}$	$[\theta]_{col}$
$[\theta]_{col}^{inv}$	$[\theta]_{row}$

Next goal: describe the matrices rep x, y, z rel 12 bases

Recall $\mathrm{Mat}_{d+1}(\mathbb{F})$: rows/cols indexed by $0, 1, \ldots, d$

Given basis $\{v_i\}_{i=0}^d$ for V. Given $A \in \mathrm{End}(V)$ and $B \in \mathrm{Mat}_{d+1}(\mathbb{F})$, say B <u>represents</u> A rel $\{v_i\}_{i=0}^d$ whenever

$$A v_j = \sum_{i=0}^{d} B_{ij} v_i \qquad 0 \le j \le d$$

Note

Given basis $\{u_i\}_{i=0}^d$ for V

-- dual basis $\{v_i\}_{i=0}^d$ for V^*

Pick $A \in \mathrm{End}(V)$

Recall $A^{adj} \in \mathrm{End}(V^*)$. The following are transposes of each other

(i) Matrix rep A rel $\{u_i\}_{i=0}^d$

(ii) -- A^{adj} rel $\{v_i\}_{i=0}^d$

LEM 64 Given basis $\{u_i\}_{i=0}^{d}$ for V, consider dual basis $\{v_i\}_{i=0}^{d}$ for V^*. The following are transposes for all $\theta \in U_q$

(i) Matrix rep θ rel $\{u_i\}_{i=0}^{d}$

(ii) ... θ^t .. $\{v_i\}_{i=0}^{d}$

$\square$

pf Recall θ^t acts on V^* as the adjoint of θ.

We define some matrices in $Mat_{d+1}(\mathbb{F})$

K_q is diagonal with (i,i)-entry q^{d-2i} for $0 \le i \le d$.

Z has (i,j)-entry $\delta_{i+j,d}$ for $0 \le i,j \le d$

so $Z^2 = I$

ex for $d=3$

$$Z = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}$$