Lecture 35 Wed Dec 4 12/4/13

1

Recall $U_q = U_q(sl_2)$

U_q-module $V = V_{d,1}$

dual space V^* is $U_{q^{-1}}$-module.

LEM 43 Let $\{V_i\}_{i=0}^d$ denote a decomp for V or V^*.

Then $\{V_i\}_{i=0}^d$ is equal to $[y]$ iff both

(i) n_x is raising for $\{V_i\}_{i=0}^d$

(ii) n_z is lowering for $\{V_i\}_{i=0}^d$

pf Assume (i), (ii) then

$n_z V_0 = 0$ $\qquad n_x V_i \subseteq V_{i+1} \quad 0 \le i \le d-1$

and

$n_x V_d = 0$ $\qquad n_z V_i \subseteq V_{i-1} \quad 1 \le i \le d$

Now $\{V_i\}_{i=0}^d$ is $[y]$ by L42 (i), (ii), (iv).

Converse is similar $\square$

LEM 44 Let $\{V_i\}_{i=0}^d$ denote a decomp of V or V^*. Then $\{V_i\}_{i=0}^d$ is equal to $[y]$ iff all

(i) x is quasi-lowering on $\{V_i\}_{i=0}^d$

(ii) y is diagonal on ...

(iii) z is quasi-raising on ...

pf $\Rightarrow$ By Th 39 or 40

$\Leftarrow$ Use L42(i),(iii)

Subspace V_0 is inv under x, y

n_z is sc mult of $1 - xy$

V_0 is inv under n_z

n_z is nilp and $\dim V_0 = 1$ so

$n_z V_0 = 0$

Sim $n_x V_d = 0$

For $0 \le i \le d-1$,

$z V_i \subseteq V_i + V_{i+1}$, $\quad y V_i \subseteq V_i$, $\quad y V_{i+1} \subseteq V_{i+1}$

So $y z V_i \subseteq V_i + V_{i+1}$

$n_x =$ sc mult of $1 - yz$

$n_x V_i \subseteq V_i + V_{i+1}$

A_x nilp on V, $\quad \dim V_q = 1 \quad 0 \le q \le d$

so

$$A_x V_i \subseteq V_{i+1} \qquad 0 \le i \le d-1$$

Now by L42 (i), (ii)

$\{V_i\}_{i=0}^d$ is $[y]$. $\square$

<u>Notation</u> (focus on V, similar for V^*)

A <u>flag</u> on V is a sequence of subspaces $\{U_i\}_{i=0}^d$ of V s.t. $U_{i-1} \subseteq U_i$ for $1 \le i \le d$ and $\dim U_i = i+1$ for $0 \le i \le d$. So $U_d = V$

Given decomp $\{V_i\}_{i=0}^d$ of V, define

$$U_i = V_0 + \cdots + V_i \qquad 0 \le i \le d.$$

Then $\{U_i\}_{i=0}^d$ is flag on V. said to be <u>induced</u> by $\{V_i\}_{i=0}^d$

Given two flags $\{U_i\}_{i=0}^d$ and $\{U_i'\}_{i=0}^d$ on V.

Call them <u>opposite</u> whenever

$$U_i \cap U_j' = 0 \quad \text{if } i+j < d \qquad (0 \le i,j \le d)$$

The above flags are opposite iff $\exists$ decomp $\{V_i\}_{i=0}^d$ of V that induces $\{U_i\}_{i=0}^d$ and whose inversion $\{V_{d-i}\}_{i=0}^d$ induces $\{U_i'\}_{i=0}^d$

In this case

$$V_i = U_i \cap U_{d-i}' \qquad 0 \le i \le d$$

LEM 45 For $0 \le i \le d+1$,

$\Lambda_x^i V$ is the sum of components $i, i+1, \ldots, d$ of decomp [y]

and the sum of components $0, 1, \ldots, d-i$ of decomp [z].

$$(+ \ CP)$$

pf Let $\{V_j\}_{j=0}^d$ denote decomp [y]

$$V = \sum_{j=0}^{d} V_j \quad \text{ds}$$

By th 38

$$\Lambda_x V_j = V_{j+1} \qquad 0 \le j \le d \qquad V_{d+1} = 0$$

So

$$\Lambda_x^i V = V_i + V_{i+1} + \cdots + V_d$$

Last assertion sim. □

COR 46 $\quad \Lambda_x^i V$ has dim $d-i+1$ for $0 \le i \le d$. $\quad (+\ CP)$

pf By L45 $\square$

COR 47 $\quad$ the sequence

$$\{\Lambda_x^{d-i} V\}_{i=0}^{d}$$

is a flag on V $\quad (+\ CP)$

pf By Cor 46 + constr. $\square$

LEM 48 In each row of the table below, we give a decomp of V and the induced flag on V.

decomp of V	induced flag on V
$[x]$	$\{n_y^{d-i} V\}_{i=0}^{d}$
$[x]^{inv}$	$\{n_z^{d-i} V\}_{i=0}^{d}$

(+ CP)

pf L45 $\square$

LEM 49 the flags

$$\{\Lambda_x^{d-i}V\}_{i=0}^{d}, \qquad \{\Lambda_y^{d-i}V\}_{i=0}^{d}, \qquad \{\Lambda_z^{d-i}V\}_{i=0}^{d}$$

are mutually opposite.

pf. By L48 and def of opposite flags. □

LEM 50 For each row of the table below we give a decomp of V along with its ith component for $0 \le i \le d$.

decomp of V	ith comp
$[x]$	$\Lambda_y^{d-i}V \cap \Lambda_z^{i}V$
$[x]^{inv}$	$\Lambda_y^{i}V \cap \Lambda_z^{d-i}V$

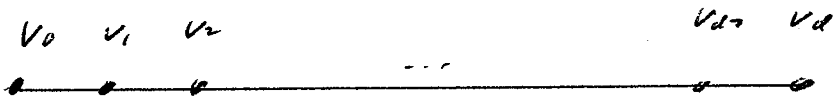
$(+\ CP)$

pf use L45 □

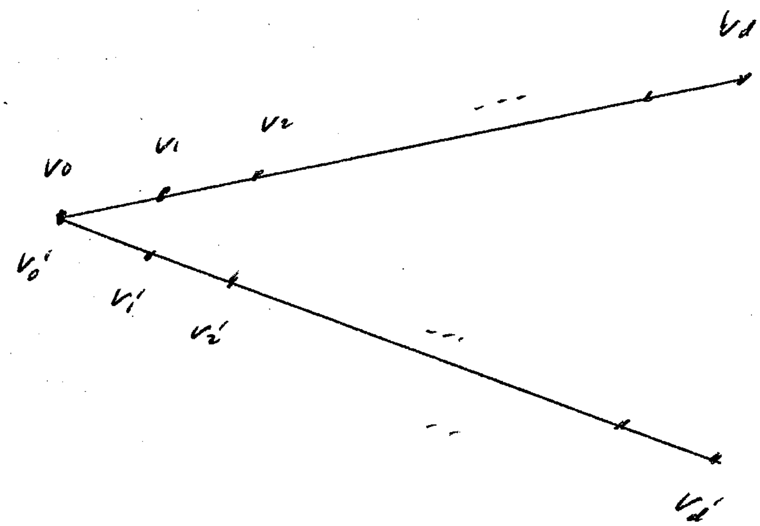
Diagram

Given decomp of V: $\{V_i\}_{i=0}^{d}$

Represent by line segment



Given 2nd decomp of V: $\{V_i'\}_{i=0}^{d}$



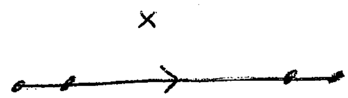
means

$$V_0 + \cdots + V_i = V_0' + \cdots + V_i' \qquad 0 \le i \le d$$

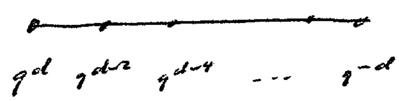
ie flag induced by $\{V_i\}_{i=0}^{d}$

$=$ " " " $\{V_i'\}_{i=0}^{d}$

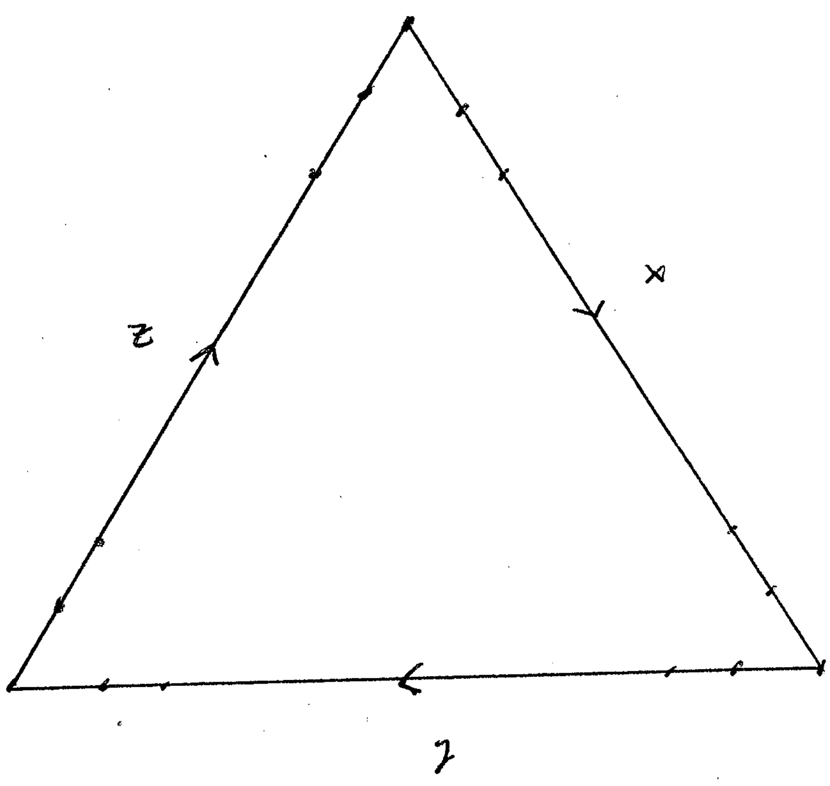
the diagram



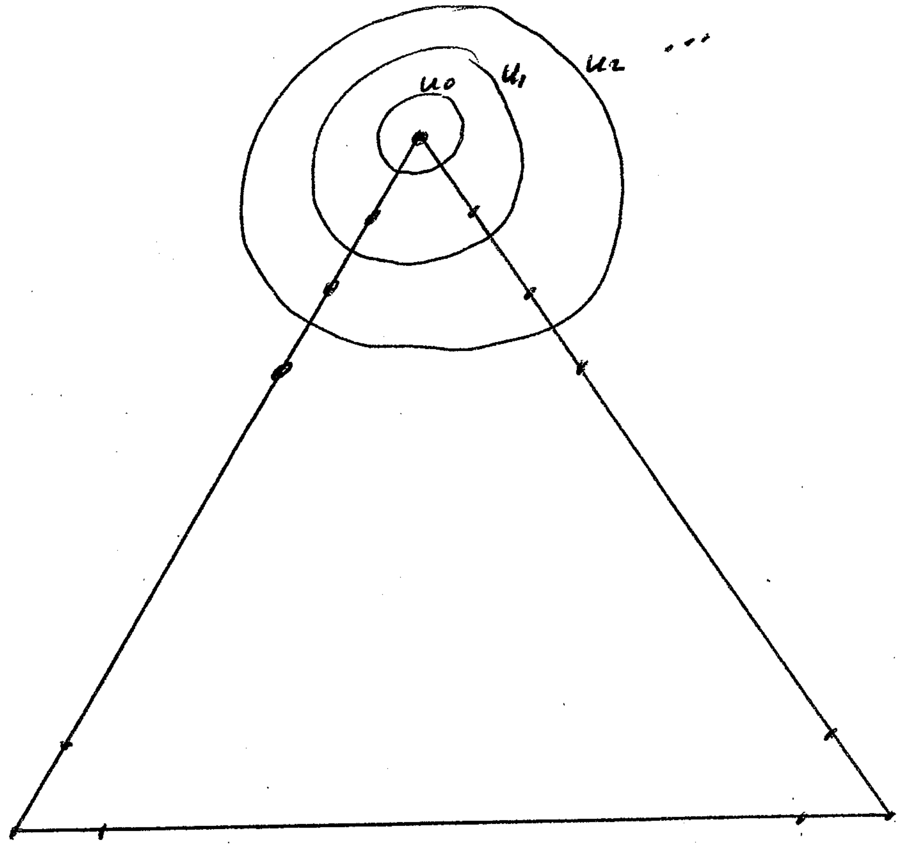
means: displayed decomp is eigenspace decomp of x, with eigenvalues



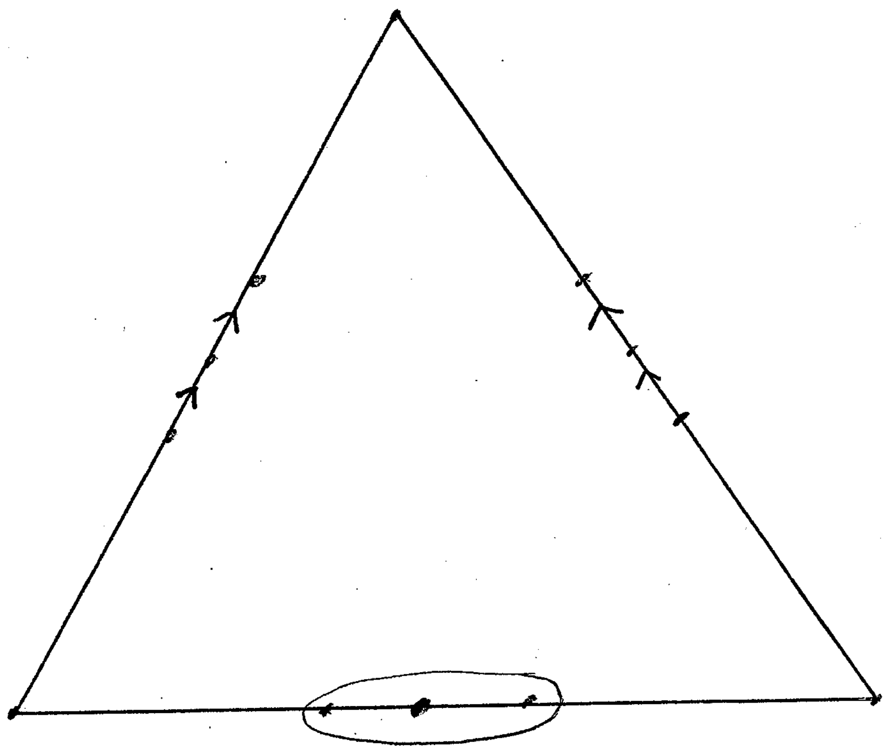
The U_q-module V:



the flag $\{\underbrace{n_y^{d-i}V}_{\substack{\shortparallel \\ U_i}}\}_{i=0}^{d}$



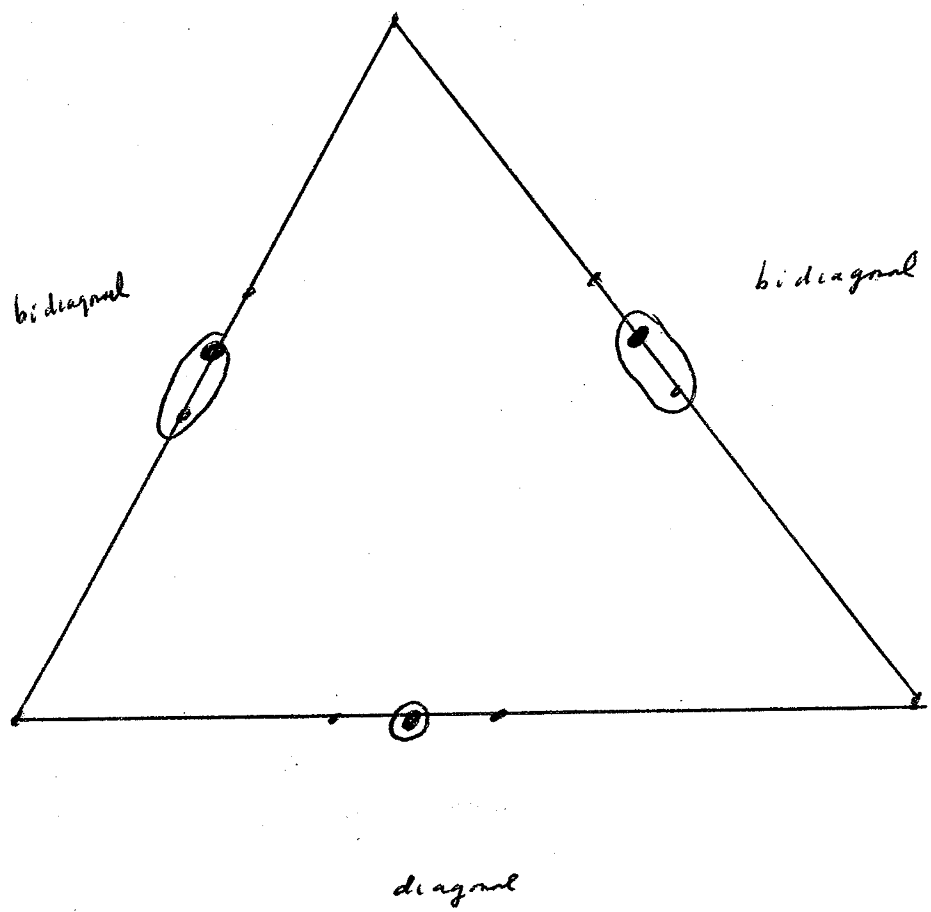
Action of A_y on V:



tridiagonal

12/4/13

the action of γ on V:



LEM 51 For $0 \le i \le d+1$,

$n_x^i V$ is the unique $(d-i+1)$-dim'l subspace of V

that is inv under y, z.

pf $n_x^i V$ has desired features by above disc.

Now let W = $(d-i+1)$-dim'l subspace of V that is inv under y, z.

show $W = n_x^i V$

Case $i = d+1$:

$W = 0 = n_x^{d+1} V$ ✓

Case $0 \le i \le d$:

$W \ne 0$

let $\{V_j\}_{j=0}^{d}$ = decomp $[y]$ of V.

y is diagonalizable on V, W is y-inv

y - - - on W

$$W = \sum_{j \in S} V_j \qquad S = \{j \mid 0 \le j \le d,\ V_j \subseteq W\}$$

$S \ne \phi$ since $W \ne 0$

n_x = sc mult of $1 - yz$

W is inv under y, z

- - - n_x

$n_x V_j = V_{j+1}$ $0 \le j \le d-1$

$j \in S \to j+1 \in S$ $0 \le j \le d-1$

$\exists\, t \quad (0 \le t \le d) \quad$ s.t

$$S = \{t, t+1, \dots, d\}$$

$$W = \sum_{j=t}^{d} V_j$$

$$\dim W = d-t+1$$

$$\| $$

$$d-i+1$$

$$t = i$$

$$W = \sum_{j=i}^{d} V_j$$

$$= \wedge_x^i V$$

$\square$