LECTURE 34 WED NOV 27 11/27/13

1

Recall $U_q = U_q(sl_2)$

U_q-module $V = V_{d,1}$

dual sp V^* is U_q-module.

Next goal: show this module is irred of type 1.

For $W \subseteq V$ define $W^\perp = \{v \in V^* \mid \langle w, v\rangle = 0 \ \forall w \in W\}$

$W \subseteq V^*$ .. V $\langle v, w\rangle = 0$ ..

Note $(W^\perp)^\perp = W$

Also $\dim W + \dim W^\perp = \dim V = \dim V^* = d+1$

LEM 32 For a subspace $U \subseteq V$ and $\theta \in U_q$

U is θ-inv iff $U^\perp$ is θ^t-inv

pf Routine $\square$

LEM 33 The U_q-module V^* is irred.

pf Use L32 and recall V is irred as U_q-module. $\square$

LEM 34 For $\theta \in U_?$ the following coincide

(i) the min poly of action of θ on V

(ii) - - - θ^{+} on V^*

pf $\forall f \in \mathbb{F}[x]$

$$f(\theta) = 0 \text{ on } V$$

$$\iff (f(\theta)u, v) = 0 \quad \forall u \in V \quad \forall v \in V^*$$

$$\shortparallel$$

$$(u, f(\theta^{+})v)$$

$$\iff f(\theta^{+}) = 0 \text{ on } V^*$$

Result follows. $\square$

LEM 35 For each of x, y, z the action on V^* is mult free with eigvals $\{q^{d-2i}\}_{i=0}^{d}$.
Moreover the $U_{q^{-1}}$-module V^* has type 1.

pf Recall each of x, y, z is mult free on V with eigvals $\{q^{d-2i}\}_{i=0}^{d}$.

Now use L34. $\square$

— o —

We now 6 decomps of V and V^* called

$$\left.\begin{matrix} [x], & [y], & [z] \\ [x]^{inv}, & [y]^{inv}, & [z]^{inv} \end{matrix}\right\} \star$$

DEF 36 For $\theta \in \{x, y, z\}$ define decomp $[\theta]$ of V (resp V^*) as follows.
For $0 \le i \le d$ the ith component of $[\theta]$ is the eigenspace for θ with eigval q^{d-2i} (resp q^{2i-d})

The inversion of $[\theta]$ is denoted $[\theta]^{inv}$

Given any decomp $\{V_i\}_{i=0}^{d}$ of V

- - - - $\{V_i'\}_{i=0}^{d}$ of V^*

Call these decomps <u>dual</u> whenever

$$(V_i, V_j') = 0 \quad \text{if} \quad i \neq j \qquad (0 \leq i, j \leq d)$$

Each decomp of V (resp V^*) is dual to a unique decomp of V^* (resp V).

LEM 37 In the table below,

for each row the given decomps are dual

dec. of V	dec. of V^*
$[x]$	$[x]^{inv}$
$[x]^{inv}$	$[x]$
$[y]$	$[y]^{inv}$
$[y]^{inv}$	$[y]$
$[z]$	$[z]^{inv}$
$[z]^{inv}$	$[z]$

pf Consider row 1

Pick i, j $(0 \le i, j \le d)$ $i \ne j$

Let $u \in$ comp i of $[x]$ on V

$v \in$ comp j of $[x]^{inv}$ on V^*

show $(u,v) = 0$

Recall

$$\begin{array}{ccc} (xu, v) & = & (u, xv) \\ \| & & \| \\ q^{d-2i}(u,v) & & (q^{-1})^{2j-d}(u,v) \\ & & \| \\ & & q^{d-2j} \end{array}$$

$i \ne j$ and q not root of 1 so $(u,v) = 0$ $\square$

Th 38 Let $\{V_i\}_{i=0}^{d}$ denote decomp

$[y]$ a $[y]^{inv}$ on V a V^*

Then for $0 \le i \le d$ the action of n_x, n_y, n_z on V_i is:

$\{V_i\}_{i=0}^{d}$	action of n_x on V_i	action of n_y on V_i	action of n_z on V_i
$[y]$	$n_x V_i = V_{i+1}$	$n_y V_i \subseteq V_{i-1}+V_i+V_{i+1}$	$n_z V_i = V_{i+1}$
$[y]^{inv}$	$n_x V_i = V_{i-1}$	$n_y V_i \subseteq V_{i-1}+V_i+V_{i+1}$	$n_z V_i = V_{i-1}$

$(+ \, CP)$

pf Using $y n_x = q^{-2} n_x y$ gives $n_x V_i \subseteq V_{i+1}$

.. $y n_z = q^2 n_z y$ .. $n_z V_i \subseteq V_{i+1}$

Show $n_x V_i = V_{i+1}$. Suppose $n_x V_i \neq V_{i+1}$

then $i \le d-1$ since $V_{d+1} = 0$

Now $\dim V_{i+1} = 1$ so $n_x V_i = 0$

Now $V_0 + \cdots + V_i$ is invariant under n_x, y, n_z

But n_x, y, n_z gen U_q so $V_0 + \cdots + V_i$ is U_q-submodule

thus submodule is nonzero and proper, cont. So $n_x V_i = V_{i+1}$

Sim $n_z V_i = V_{i+1}$

Show $n_y V_i \le V_{i-1} + V_i + V_{i+1}$

$$n_y = \frac{q(1-zx)}{q-q^{-1}} \qquad (*)$$

$$n_x = \frac{q(1-yz)}{q-q^{-1}}$$

so

$$z = y\left(1 - q^{-1}(q-q^{-1})\, n_x\right)$$

so

$$zV_i \le V_i + V_{i+1}$$

$$n_z = \frac{q(1-xy)}{q-q^{-1}}$$

so

$$x = \left(1 - q^{-1}(q-q^{-1})\, n_z\right) y^{-1}$$

so

$$xV_i \le V_i + V_{i-1}$$

Now using $(*)$

$$n_y V_i \le V_{i-1} + V_i + V_{i+1}$$

$\square$

Th 39 Let $\{V_i\}_{i=0}^{d}$ denote the decomp $[y]$ or $[y]^{inv}$ for V. Then for $0 \le i \le d$ the action of x, y, z on V_i is:

$\{V_i\}_{i=0}^{d}$	action of x on V_i	action of y on V_i	action of z on V_i
$[y]$	$(x - q^{2i-d}I)V_i = V_{i-1}$	$(y - q^{d-2i}I)V_i = 0$	$(z - q^{2i-d}I)V_i = V_{i+1}$
$[y]^{inv}$	$(x - q^{d-2i}I)V_i = V_{i+1}$	$(y - q^{2i-d}I)V_i = 0$	$(z - q^{d-2i}I)V_i = V_{i-1}$

(+ CP)

pf $[y]$: Using Th 38 and

$$x = y^{-1} - q^{-1}(q - q^{-1})\, n_z y^{-1}$$

$$z = y^{-1} - q(q - q^{-1})\, n_x y^{-1}$$

$$(x - q^{2i-d}I)V_i = (x - y^{-1})V_i = n_z y^{-1} V_i = n_z V_i = V_{i-1}$$

$$(z - q^{2i-d}I)V_i = (z - y^{-1})V_i = n_x y^{-1} V_i = n_x V_i = V_{i+1}$$

$[y]^{inv}$: clear

$\square$

Th 40 Let $\{V_i\}_{i=0}^d$ denote the decomp $[y]$ or $[y]^{inv}$ for V. Then for $0 \le i \le d$ the action of x, y, z on V_i is

$\{V_i\}_{i=0}^d$	action of x on V_i	action of y on V_i	action of z on V_i
$[y]$	$(x - q^{d-2i} I) V_i = V_{i+1}$	$(y - q^{2i-d} I) V_i = 0$	$(z - q^{d-2i} I) V_i = V_{i+1}$
$[y]^{inv}$	$(x - q^{2i-d} I) V_i = V_{i+1}$	$(y - q^{d-2i} I) V_i = 0$	$(z - q^{2i-d} I) V_i = V_{i+1}$

pf In Th 39 replace q by q^{-1} □

LEM 41 Referring to V or V^* the following coincide for $0 \le i \le d$

(i) the ith component of the dec $[y]$

(ii) $n_x^i \ker(n_z)$

(iii) $n_z^{d-i} \ker(n_x)$

pf Use Th 38 $\square$

LEM 42 Let $\{V_i\}_{i=0}^d$ denote a decomp of V rel V_0^*. TFAE:

(i) $\{V_i\}_{i=0}^d$ is equal to $[y]$

(ii) $n_z V_0 = 0$ and $n_x V_i \subseteq V_{i+1}$ for $0 \le i \le d-1$

(iii) $n_z V_0 = 0$ and $n_x^i V_0 \subseteq V_i$ for $0 \le i \le d$

(iv) $n_x V_d = 0$ and $n_z V_i \subseteq V_{i-1}$ for $1 \le i \le d$

(v) $n_x V_d = 0$ and $n_z^{d-i} V_d \subseteq V_i$ for $0 \le i \le d$.

$\square$

pf Routine.