LECTURE 33 Monday Nov 25

Recall assumptions

$\mathbb{F}$ arb

$0 \neq q \in \mathbb{F}$ not a root of 1

$U_q = U_q(sl_2)$

Chev gens e, f, k, k^{-1}

equit gens x, y, y^{-1}, z

We saw that Ω^{-1} is a rotator for U_q, where

$$\Omega = \exp_q(nx)\, \mathcal{I}\, \exp_q(nz)$$

Next goal: relate Ω to the Lusztig operators.

Recall the U_q-module $V_{d,\varepsilon}$

Abbr $V_d = V_{d,1}$

DEF 20 For $V = V_d$ we define

$\tau_x, \tau_y, \tau_z \in \mathrm{End}(V)$

by

$$\tau_x = \exp_q(ny)\,\Omega$$

$$\tau_y = \exp_q(nz)\,\Omega$$

$$\tau_z = \exp_q(nx)\,\Omega$$

On V_d,

$$\Omega^{-1} T_x \Omega = T_y$$

$$\Omega^{-1} T_y \Omega = T_z$$

$$\Omega^{-1} T_z \Omega = T_x$$

Abbr

$$N_x = \exp_q(n_x)$$

$$N_y = \exp_q(n_y)$$

$$N_z = \exp_q(n_z)$$

LEM 21 On the U_q-module V_d,

(i) $T_y^{-1} y T_y = y^{-1}$

(ii) $T_y^{-1} n_z T_y = n_x$

(iii) $T_y^{-1} n_x T_y = y^{-1} n_z y^{-1}$

pf (i) $T_y^{-1} y T_y = \Omega^{-1} \underbrace{N_z^{-1} y N_z}_{= x^{-1}} \Omega$

$= y^{-1}$

(ii) $T_y^{-1} n_z T_y = \Omega^{-1} \underbrace{N_z^{-1} n_z N_z}_{= n_z} \Omega$

$= n_x$

(iii) $T_y^{-1} n_x T_y = \Omega^{-1} \underbrace{N_z^{-1} n_x N_z}_{= x^{-1} n_y x^{-1}} \Omega$

$= y^{-1} n_z y^{-1}$ $\square$

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LEM 22 On the U_q-module V_d

$$\mathcal{Y} = \Omega^3 \, T_y^{-2} \qquad (+CP)$$

pf

$$\mathcal{Y} T_y^2 = \underbrace{\mathcal{Y} N_z}_{= N_z \mathcal{X} \text{ (L19)}} \underbrace{\Omega N_z}_{= N_y \Omega} \Omega$$

$$= \underbrace{N_z \mathcal{X} N_y}_{= \Omega \text{ (L18)}} \Omega^2$$

$$= \Omega^3$$

$\square$

We now describe the action of T_y on the U_q-module $V = V_d$.

For $\lambda \in \mathbb{Z}$ define

$$V_\lambda = \{ v \in V \mid kv = q^\lambda v \}$$

$V_\lambda \neq 0$ iff $\exists\, i \ (0 \leq i \leq d)$ s.t. $\lambda = d - 2i$

In this case $\dim V_\lambda = 1$

Call V_λ the <u>λ-weight space</u> whenever $V_\lambda \neq 0$

We have
$$V = \sum_{i=0}^{d} V_{d-2i} \qquad \text{(ds)}$$

For $\lambda \in \mathbb{Z}$

$$e V_\lambda \subseteq V_{\lambda+2}$$

$$f V_\lambda \subseteq V_{\lambda-2}$$

By L 21(i)
$$T_y V_\lambda = V_{-\lambda}$$

Thm 2.3 With the above notation, the following holds on each weight space V_λ:

$$(i) \quad T_y = q^{\frac{d^2+\lambda}{2}} \sum_{\substack{a,b,c \geq 0 \\ b-a-c=\lambda}} \frac{e^a f^b e^c}{[a]_q^! [b]_q^! [c]_q^!} q^{ac-b} (-1)^b$$

$$(ii) \quad T_y = (-1)^d q^{\frac{d^2-\lambda}{2}} \sum_{\substack{a,b,c \geq 0 \\ a+c-b=\lambda}} \frac{f^a e^b f^c}{[a]_q^! [b]_q^! [c]_q^!} q^{ac-b} (-1)^b$$

$$(iii) \quad T_y^{-1} = (-1)^d q^{-\frac{d^2+\lambda}{2}} \sum_{\substack{a,b,c \geq 0 \\ b-a-c=\lambda}} \frac{e^a f^b e^c}{[a]_q^! [b]_q^! [c]_q^!} q^{b-ac} (-1)^b$$

$$(iv) \quad T_y^{-1} = q^{\frac{\lambda-d^2}{2}} \sum_{\substack{a,b,c \geq 0 \\ a+c-b=\lambda}} \frac{f^a e^b f^c}{[a]_q^! [b]_q^! [c]_q^!} q^{b-ac} (-1)^b$$

[The above sums are the Lusztig operators. See Jantzen Lectures on quantum groups Ch 8] pf (i) Identify

$k = y \qquad e = n_z \qquad f = -q^{-1} y^{-1} n_x$

$$n_x = -qyf$$

$$T_y = N_z \Omega$$

$$= N_z \, N_x \, Y \, N_z$$

$$= N_z \, N_x \, Y N_z Y^{-1} \, Y$$

$$= N_z \, N_x \, \exp_q\Big(\underbrace{Y n_z Y^{-1}}\Big) \, Y$$

$$= y^{-1} n_z y^{-1}$$

$$= y^{-2} n_z q^{2}$$

[write $\lambda = d - 2l$]

On V_λ

$$T_y = \sum_{a\geq 0} \sum_{b \geq 0} \sum_{c \geq 0} \frac{q^{\binom{a}{2}} q^{\binom{b}{2}} q^{\binom{c}{2}}}{[a]_q^! \, [b]_q^! \, [c]_q^!} \, n_z^a \, n_x^b \, (y^{-2} n_z q^2)^c \, q^{2l(d-}$$

☆

write RHS in terms of e, f

$$n_z^a = e^a$$

$$n_x^b = (-qyf)^b$$

$$= (-1)^b f^b y^b q^{-b^2}$$

$$\left(y^{-2} n_z q^2\right)^c = e^c y^{-2c} q^{-2c^2}$$

In the sum ✱ need only consider the terms

such that $b-a-c=\lambda$

Reason: we saw $T_y V_\lambda = V_{-\lambda}$

Also $e^a f^b e^c \, V_\lambda \subseteq V_{-\lambda}$ iff $b-a-c=\lambda$

Result follows after routine reduction

(ii) $$\begin{aligned} T_y &= N_z \Omega \\ &= \Omega N_x \\ &= N_x \Upsilon N_z N_x \\ &= N_x \underline{\Upsilon N_z \Upsilon^{-1}} \, \underline{\Upsilon N_x \Upsilon^{-1}} \Upsilon \\ &= N_x \exp_q\big(\underbrace{\Upsilon n_z \Upsilon^{-1}}_{= y^{-2} n_z q^2}\big) \exp_q\big(\underbrace{\Upsilon n_x \Upsilon^{-1}}_{= y n_x y = y^2 n_x q^2}\big) \Upsilon \end{aligned}$$

Now continue as in part (i)

(iii), (iv) Sim

∎

Next goal: compare

$$U_q(sl_2), \quad U_{q^{-1}}(sl_2)$$

For both algebras use same notation x, y, z for equit gens

LEM 24. The equit presentation for $U_{q^{-1}}(sl_2)$ has generators x, y, y^{-1}, z and rels

$$y^{-1}y = 1, \quad yy^{-1} = 1,$$

$$\frac{qzy - q^{-1}yz}{q - q^{-1}} = 1$$

$$\frac{qyx - q^{-1}xy}{q - q^{-1}} = 1$$

$$\frac{qxz - q^{-1}zx}{q - q^{-1}} = 1$$

pf In the equit pres of $U_q(sl_2)$ replace q by q^{-1} and rearrange terms. $\square$

LEM 25 $\exists$ $\mathbb{F}$-alg iso $U_q(sl_2) \to U_{q^{-1}}(sl_2)$

that sends

$$x \to z, \quad y \to y, \quad z \to x.$$

pf Compare L24 with the equit pres of $U_q sl_2$ $\square$

Prop 26 $\exists$ anti isomorphism of $\mathbb{F}$-algebras

$$\dagger : U_q(sl_2) \longrightarrow U_{q^{-1}}(sl_2)$$

that sends

$$x \to x, \quad y \to y, \quad z \to z$$

pf Use L24 Recall $\dagger$ swaps order of mult. $\square$

LEM 27 the anti iso $\dagger$ sends

$$n_x \to -n_x, \quad n_y \to -n_y, \quad n_z \to -n_z$$

pf In $U_q(sl_2)$

$$n_x = \frac{q(1-yz)}{q-q^{-1}} = \frac{q^{-1}(1-zy)}{q-q^{-1}}$$

In $U_{q^{-1}}(sl_2)$

$$n_x = \frac{q^{-1}(1-yz)}{q^{-1}-q} = \frac{q(1-zy)}{q^{-1}-q}$$

So

$$n_x = \frac{q(1-yz)}{q-q^{-1}} \xrightarrow{T} \frac{q(1-zy)}{q-q^{-1}} = -n_x$$

$\square$

For the $U_q(sl_2)$-module $V = V_d$

we now turn the dual space V^* into a module

for $U_{q^{-1}}(sl_2)$

Recall

$$V^* = \{ f \mid f : V \to \mathbb{F} \text{ is } \mathbb{F}\text{-linear} \}$$

V^* is vector space $/ \mathbb{F}$ with dim $d+1$

DEF 28 We define a bilinear form

$$(\, , \,) : V \times V^* \to \mathbb{F}$$

s.t. $(u, f) = f(u)$ $\quad \forall u \in V \quad \forall f \in V^*$

The form is nondegenerate.

For $A \in End(V)$,

the <u>adjoint</u> of A, denoted A^{adj}

is unique element of V^* such that

$$(Au, v) = (u, A^{adj} v) \quad \forall u \in V \quad \forall v \in V^*$$

The map

$$adj: \begin{array}{ccc} End(V) & \to & End(V^*) \\ A & \to & A^{adj} \end{array}$$

is an anti isomorphism of $\mathbb{F}$- algebras.

Prop 29 With above notation,

$\exists$ unique $U_{q^{-1}}(sl_2)$-module str on V^* s.t.

$$(Au, v) = (u, A^{\dagger}v) \quad \forall u \in V,\ \forall v \in V^* \quad \forall A \in U_q(sl_2)$$

pf Action of $U_q(sl_2)$ on V induces $\mathbb{F}$-alg hom

$$U_q(sl_2) \to End(V)$$

Call this map ψ.

The composition

$$U_{q^{-1}}(sl_2) \xrightarrow{\dagger^{-1}} U_q(sl_2) \xrightarrow{\psi} End(V) \xrightarrow{adj} End(V^*)$$

is $\mathbb{F}$-alg hom. This gives V^* a module str for $U_{q^{-1}}(sl_2)$.

One checks this module has desired features, and is unique. $\square$

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Prop 30 For $A \in U_q(sl_2)$, $A^\dagger$ acts on V^* as the adjoint of the A action on V.

pf By Prop 29 $\square$

Prop 31 For $u \in V$ and $v \in V^*$

$$(xu, v) = (u, xv) \qquad (yu, v) = (u, yv), \qquad (zu, v) = (u, zv)$$

$$(\eta_x u, v) = -(u, \eta_x v) \qquad (\eta_y u, v) = -(u, \eta_y v) \qquad (\eta_z u, v) = -(u, \eta_z v)$$

pf By Prop 26, Lem 27. $\square$