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LEM 3 On the U_q-module $V_{d,\varepsilon}$

(i) Each of x, y, z is mult free with eigenvalues

$$\{q^{d-2i}\}_{i=0}^{d}$$

(ii) Each of x, y, z is invertible

(iii) Each of n_x, n_y, n_z is nilpotent

pf (i) By L2

(ii) By (i)

(iii) Recall $n_x y = q^2 y n_x$

So $n_x^{d+1} = 0$ on $V_{d,\varepsilon}$ $\square$

Sim for n_y, n_z

— o —

We now consider

$\exp_q(n_x)$, $\exp_q(n_y)$, $\exp_q(n_z)$

On the U_q-module $V_{d,\varepsilon}$

of course

$$\exp_q(\Lambda z)^{-1} \ \Lambda \ \exp_q(\Lambda z) = \Lambda$$

$$\exp_q(\Lambda z)^{-1} \ \Lambda z \ \exp_q(\Lambda z) = \Lambda z$$

$(+CP)$

LEM 4 On the U_q-module $V_{d,\varepsilon}$

$$\exp_q(\Lambda z)^{-1}\, y\, \exp_q(\Lambda z) = x^{-1} \qquad (+CP)$$

pf Show

$$y \exp_q(\Lambda z) x \overset{?}{=} \exp_q(\Lambda z)$$

$$\|$$

$$y x\, x^{-1} \exp_q(\Lambda z) x$$

$$\|$$

$$\underbrace{y x}_{=\,1-(q^2-1)\Lambda z} \exp_q(\underbrace{x^{-1} \Lambda z x}_{=\,q^2 \Lambda z})$$

Require

$$\left(1-(q^2-1)\Lambda z\right) \exp_q(\Lambda z\, q^2) = \exp_q(\Lambda z)$$

This is L110

□

LEM 5 For $i \geq 0$

$$z\, n_z^i - n_z^i\, z = [i]_q\, q^{1-i} \left(n_z^{i-1} x - y\, n_z^{i-1} \right)$$

pf Ind on i (Routine) $\square$

LEM 6 On the U_q-module $V_{d,\varepsilon}$

$$(y+z)\exp_q(n_z) = \exp_q(n_z)(x+z) \qquad (+\,cp)$$

pf On each side eval using

$$\exp_q(n_z) = \sum_{i=0}^{\infty} \frac{q^{\binom{i}{2}} n_z^i}{[i]_q^!}$$

and use L $\square$

LEM 7 On the U_q-module $V_{d,\varepsilon}$

$$\exp_q(\Lambda z)^{-1}\, z\, \exp_q(\Lambda z) = z + x - x^{-1} \qquad (+\,CP)$$

pf

$$LHS = \exp_q(\Lambda z)^{-1}(y+z)\exp_q(\Lambda z)$$

$$- \exp_q(\Lambda z)^{-1}\, y\, \exp(\Lambda z)$$

$$= z + x \quad - x^{-1}$$

□

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LEM 8 On the U_q-module $V_{d,\varepsilon}$

$$\exp_q(nz)^{-1}\; x \;\exp_q(nz) = xyx \qquad (+CP)$$

pf Obs.

$$\exp_q(nz)^{-1}\; nz\; \exp_q(nz) = nz$$

Eval this using $\quad nz = \frac{q(1-xy)}{q-q^{-1}}$

and L4 $\square$

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LEM 9 On the U_q-module $V_{d,\varepsilon}$

$$\exp_q(\eta z)^{-1}\, \eta_x\, \exp_q(\eta_z) = X^{-1} \eta_y X^{-1} \qquad (+\,CP)$$

Pf

$$(q-q^{-1})\,LHS = q \exp_q(\eta z)^{-1}(1-yz)\exp_q(\eta_z)$$

$$= q\left(1 - x^{-1}(z+x-x^{-1})\right)$$

$$= q\,(x^{-2} - x^{-1}z)$$

$$= x^{-2}q^{2}(1-xz)$$

$$= X^{-2}q^{2}\eta_y\,(q-q^{-1})$$

$$= x^{-1}\eta_y X^{-1}(q-q^{-1})$$

$\square$

LEM 10 On the U_q-module $V_{d,\varepsilon}$

$$\exp_q(\Lambda z)^{-1}\ \Lambda y\ \exp_q(\Lambda z) =$$

$$\frac{\Lambda x}{q-q^{-1}} + \Lambda y - \frac{q+q^{-1}}{q-q^{-1}} x^2 + x \Lambda z X$$

(+CP)

Pf Recall

$$\nu_y y = \Lambda - qz - q^{-1}x$$

In this eq conj ~~each side by $\exp_q(\Lambda z)$~~

$\square$

DEF 11 For $V = V_{d,\varepsilon}$ by a <u>rotator</u> for V we mean an invertible $R \in \mathrm{End}(V)$ s.t

$$RxR^{-1} = y, \quad RyR^{-1} = z, \quad RzR^{-1} = x$$

LEM 12 There exists a rotator for $V_{d,\varepsilon}$

pf For $V = V_{d,\varepsilon}$ define

$$x', y', z' \in \mathrm{End}(V)$$

s.t.

map	x'	y'	z'
action on V	y	z	x

then x', y', z' satisfy the defining relations for U_q

So V supports a U_q-module structure s.t.

gen	x	y	z
action on V	x'	y'	z'

The new U_q-module V is irred, and type ε; hence iso to $V_{d,\varepsilon}$

let $R \in \mathrm{End}(V)$ denote an iso of U_q-modules from the orig U_q-module V to the new U_q-module V. R has the desired properties. $\square$

LEM 13 Let R denote a rotator for $V = V_{d,\varepsilon}$

Then for $R' \in \mathrm{End}(V)$ TFAE

(i) R' is a rotator for V

(ii) $\exists\ 0 \neq \alpha \in \mathbb{F}$ s.t. $R' = \alpha R$

pf Routine $\square$

Next goal: give an explicit rotator

DEF 14 For $V = V_{d,\varepsilon}$ define

$$\mathfrak{X},\ \mathfrak{Y},\ \mathfrak{Z} \in \mathrm{End}(V)$$

as follows.

Recall for $0 \le i \le d$ $v_i \in V$ is an eigenvector for y with eigval εq^{d-2i}.

Then v_i is eigvector for $\mathfrak{Y}$ with eigval $\varepsilon q^{2i(d-i)}$.

This defines $\mathfrak{Y}$, Define $\mathfrak{X}, \mathfrak{Z}$ similarly.

— o —

Since $\{q^{d-2i}\}_{i=0}^{d}$ are distinct, $\exists$ polynml $G \in \mathbb{F}[\lambda]$ s.t. $\deg G \le d$ and

$$G(q^{d-2i}) = q^{2i(d-i)} \qquad 0 \le i \le d$$

On $V_{d,\varepsilon}$,

$$\varepsilon \mathfrak{X} = G(\varepsilon x) = G(\varepsilon x^{-1})$$
$$\varepsilon \mathfrak{Y} = G(\varepsilon y) = G(\varepsilon y^{-1})$$
$$\varepsilon \mathfrak{Z} = G(\varepsilon z) = G(\varepsilon z^{-1})$$

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LEM 15 On the U_q-module $V_{d,\varepsilon}$

(i) $Y^{-1} y Y = y$ (HCP)

(ii) $Y^{-1} n_z Y = y n_z y$

(iii) $Y^{-1} n_x Y = y^{-1} n_x y^{-1}$

pf (i) ✓

(ii) For $0 \le i \le d$ let v_i = eigenvector for y with eigenval q^{d-2i}

$$y n_z = q^2 n_z y$$

$$\left(y - q^{d-2i+2}\right) n_z v_i = 0$$

$$Y^{-1} n_z Y v_i = \underbrace{Y^{-1} n_z v_i}_{= q^{-2(i-1)(d-i+1)} n_z v_i} q^{2i(d-i)}$$

Also $$y n_z y v_i = \underbrace{y n_z v_i}_{= q^{d-2i+2} n_z v_i} q^{d-2i}$$

$$2i(d-i) - 2(i-1)(d-i+1) \overset{?}{=} d-2i + d-2i+2$$

yes □

(iii) Sim

DEF 16 For $V = V_{d,\varepsilon}$ we define

$\Omega \in End(V)$ by

$$\Omega = \exp_q(\Lambda_x)\ \underline{Y}\ \exp_q(\Lambda_z)$$

obs Ω^{-1} exists.

Prop 17 Ω^{-1} is a rotator for $V_{d,\varepsilon}$

pf

Check

$$\Omega^{-1} \times \Omega = y$$

$$\Omega = \exp_q(n_x)\, \Psi\, \exp_q(n_z)$$

$$\exp_q(n_x)^{-1} \times \exp_q(n_x) = x + y - y^{-1}$$

$$= (xy - 1)y^{-1} + y$$

$$= y - q^{-1} V_z y^{-1}$$

$$\Psi^{-1}(y - q^{-1} V_z y^{-1})\Psi = y - q^{-1}(y V_z y) y^{-1}$$

$$= y(1 - q^{-1} V_z)$$

$$= y x y$$

$$\exp_q(n_z)^{-1}\,(y x y)\,\exp_q(n_z) = x^{-1}(x y x)\,x^{-1}$$

$$= y \quad \checkmark$$

check

$$\Omega^{-1}\, y\, \Omega = z$$

$$\Omega = \exp_q(\eta x)\ \mathcal{Y}\ \exp_q(\eta z)$$

$$\exp_q(\eta x)^{-1}\, y\, \exp_q(\eta x) = \underbrace{y z}_{1-q^{-1}\nabla_x} {}^{-1}$$

$$\mathcal{Y}^{-1}\,(y z^{-1})\,\mathcal{Y} = y - q^{-1}\,(y^{-1}\nabla_x y^{-1})\, y$$

$$= y - q^{-1}\, y^{-1}\nabla_x$$

$$= y - y^{-1} + z$$

$$\exp_q(\eta z)^{-1}\,(y - y^{-1} + z)\,\exp_q(\eta z) =$$

$$x^{-1} - x + z + x - x^{-1}$$

$$= z$$

Check

$$\Omega^{-1} z \, \Omega = x$$

$$\Omega = \exp_q(\pi x) \; Y \; \exp_q(\pi z)$$

$$\exp_q(\pi x)^{-1} \; z \; \exp_q(\pi x) = y^{-1}$$

$$Y^{-1} y^{-1} Y = y^{-1}$$

$$\exp_q(\pi z)^{-1} \; y^{-1} \; \exp_q(\pi z) = x$$

✓

□

By Prop 17 and the construction

$$\Omega^{-1} \Lambda_x \Omega = \Lambda_y$$

$$\Omega^{-1} \Lambda_y \Omega = \Lambda_z$$

$$\Omega^{-1} \Lambda_z \Omega = \Lambda_x$$

Also

$$\Omega^{-1} \Lambda \Omega = \Lambda$$

Also since $\varepsilon X = G(\varepsilon x)$ etc

$$\Omega^{-1} X \Omega = Y$$

$$\Omega^{-1} Y \Omega = Z$$

$$\Omega^{-1} Z \Omega = X$$

LEM 18 Ω is equal to each of

$$\exp_q(\Lambda_x)\, Y \exp_q(\Lambda_z)$$

$$\exp_q(\Lambda_y)\, Z \exp_q(\Lambda_x)$$

$$\exp_q(\Lambda_z)\, X \exp_q(\Lambda_y)$$

pf In the eq $\Omega = \exp_q(\Lambda_x)\, Y \exp_q(\Lambda_z)$

conjugate each side by Ω

$\square$

LEM 19 On the U_q module $V_{d,\varepsilon}$

$$\exp_q(\eta z)^{-1}\, Y \,\exp_q(\eta z) = X \qquad (+CP)$$

pf By L4

$$\exp_q(\eta z)^{-1}\, y \,\exp_q(\eta z) = x^{-1}$$

So

$$\exp_q(\eta z)^{-1}\, \underbrace{G(\varepsilon y)}_{=\varepsilon Y} \,\exp_q(\eta z) = \underbrace{G(\varepsilon x^{-1})}_{=\varepsilon X}$$

$\square$