Recall Assumptions: Given LS

$$\Phi = (A, \{E_i\}_{i=0}^d ; A^*; \{E_i^*\}_{i=0}^d)$$

on $V$ of $q$-Racah type, $d \ge 1$

$$\theta_i = a q^{2i-d} + a^{-1} q^{d-2i} \qquad 0 \le i \le d$$
$$\theta_i^* = b q^{2i-d} + b^{-1} q^{d-2i}$$

Replacing $\mathbb{F}$ by $\bar{\mathbb{F}}$ if nec, WLOG $q, a, b \in \mathbb{F}$

Recall

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad 0 \le i \le d+1$$

Split decomp $\{U_i\}_{i=0}^d$

By Th 147

$$\psi U_i \subseteq U_{i-1}$$
$$\psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad 0 \le i \le d$$

Def 148 Define $K, B \in End(V)$ s.t.

for $0 \le i \le d$,

$U_i$ = eigenspace for $K$ with eigenval $q^{d-2i}$

$U_i^{\Downarrow}$ = ... $B$ ...

So

$$(K - q^{d-2i} I) U_i = 0 \qquad 0 \le i \le d$$

$$(B - q^{d-2i} I) U_i^{\Downarrow} = 0 \qquad ''$$

LEM 149 We have

(i) $$\frac{qKA - q^{-1}AK}{q-q^{-1}} = a^{-1}K^2 + aI$$

(ii) $$\frac{qBA - q^{-1}AB}{q-q^{-1}} = aB^2 + a^{-1}I$$

(iii) $K\Psi = q^2\Psi K$, $B\Psi = q^{-2}\Psi B$

pf (i),(ii) By L126
(iii) By L116 □

thm 150 (Sarah Bockting-Conrad)

$$0 = aB^2 - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} BK - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} KB + a^{-1}K^2$$

$$0 = aK^{-2} - \frac{a^{-1}q - aq^{-1}}{q - q^{-1}} B^{-1}K^{-1} - \frac{aq - a^{-1}q^{-1}}{q - q^{-1}} K^{-1}B^{-1} + a^{-1}B^{-2}$$

pf By th 122. $\square$

DEF 151 Put

$$M = \frac{aB - a^{-1}K}{a - a^{-1}}$$

(compare with L119)

LEM 152 $M$ is diagonalizable with eigvals $\{q^{d-2i}\}_{i=0}^{d}$. Moreover $M^{-1}$ exists.

pf By L119 and Def 115 □

thm 153 We have

$$\frac{qM^{-1}K - q^{-1}KM^{-1}}{q-q^{-1}} = I,$$

$$\frac{qM^{-1}B - q^{-1}BM^{-1}}{q-q^{-1}} = I.$$

pf By th 123 □

— o —

Define

$$M' = \frac{aK^{-1} - a^{-1}B^{-1}}{a-a^{-1}}$$

[ this is obtained from $M$ by replacing $q \to q^{-1}$, $a \to a^{-1}$ ]

**LEM 154** Fix $0 \neq v \in E_0^* V$

With respect to the basis $\{T_i(A)v\}_{i=0}^{d}$

the matrices representing $M^{-1}$, $(M')^{-1}$ are upper bi-diag. with the following entries.

| matrix | $(i,i)$-entry | $(i-1,i)$-entry |
|---|---|---|
| $M^{-1}$ | $q^{2i-d}$ | $(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})\, q^{2i-d-1}\, a$ |
| $(M')^{-1}$ | $q^{d-2i}$ | $(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})\, q^{d+1-2i}\, a^{-1}$ |

pf By L 124 (i) $\square$

Prop 155

$$A^* - bM^{-1} - b^{-1}(M')^{-1} \in \mathbb{F}\Psi$$

pf For $0 \le i \le d$

$$(A^* - \theta_i^* I) U_i \subseteq U_{i-1}$$

and

$$\theta_i^* = b q^{2i-d} + b^{-1} q^{d-2i}$$

By L154

$$(M^{-1} - q^{2i-d} I) U_i \subseteq U_{i-1}$$

$$((M')^{-1} - q^{d-2i} I) U_i \subseteq U_{i-1}$$

So

$$(A^* - bM^{-1} - b^{-1}(M')^{-1}) U_i \subseteq U_{i-1} \qquad 0 \le i \le d$$

Note that $\Downarrow$ sends

$$q \to q, \quad a \to a^{-1}, \quad K \leftrightarrow B$$

and fixes

$$A^*, \ M, \ M', \ b$$

So

$$(A^* - bM^{-1} - b^{-1}(M')^{-1}) U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow} \qquad 0 \le i \le d$$

Result follows via COR 97 $\square$

DEF 156 By Prop 155

$\exists \; 0 \neq c \in \overline{\mathbb{F}}$ s.t

$$\frac{A^* - bM^{-1} - b^{-1}(M')^{-1}}{(q-q^{-1})(q^d-q^{-d})} = (c+c^{-1})I \qquad (*)$$

Scalar c is defined up to inverse.

Thm 157 For our LS $\Phi$ of $q$-Racah type, the 1st and 2nd split sequences are

$$\varphi_i = a^{-1}b^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - abc\, q^{i-d-1})(q^{-i} - abc^{-1}q^{i-d-1})$$

$$\phi_i = ab^{-1}q^{d+1}(q^i - q^{-i})(q^{i-d-1} - q^{d-i+1})(q^{-i} - a^{-1}bc\, q^{i-d-1})(q^{-i} - a^{-1}bc^{-1}q^{i-d-1})$$

for $1 \le i \le d$

pf To get $\varphi_i$, in ($\star$) apply each side to the vector

$\tau_i(A)v$ $\quad (0 \ne v \in E_0^* V)$

Note that

| action of | given in |
|---|---|
| $A^*$ | below L 145 |
| $M^{-1}$ | L 154 |
| $(M')^{-1}$ | L 154 |
| $\Psi$ | Th 147 |

To get $\phi_i$ find 1st split sequence of $\Phi^{\Downarrow}$

Recall $\Downarrow$ sends $q \to q$ $\quad a \to a^{-1}$ $\quad b \to b$ $\quad c \to c$

□

LEM 158 For the scalar $c$ in DEF 156

none of

$$abc, \quad a^{-1}bc, \quad ab^{-1}c, \quad abc^{-1}$$

is among

$$q^{d-1}, \; q^{d-3}, \; \ldots, \; q^{1-d}$$

pf By Th 157 and since $\varphi_i \neq 0$, $\phi_i \neq 0$ for $1 \leq i \leq d$ $\square$

# CHAPTER III $U_q(sl_2)$ and TD pairs

For this chapter

field $\mathbb{F}$ is arbitrary

Fix $0 \neq q \in \mathbb{F}$ $q$ not a root of 1

Recall $U_q = U_q(sl_2)$ from Def 50

Chevalley gens $e, f, k, k^{-1}$

Equitable gens $x, y, y^{-1}, z$

Casimir element

$$\Lambda = (q-q^{-1})^2 ef + q^{-1}k + qk^{-1}$$

$$= qx + q^{-1}y + qz - qxyz$$

$\Lambda$ is central

Def

$$\nu_x = q(1-yz) = q^{-1}(1-zy) = \frac{yz-zy}{q-q^{-1}}$$

$\nu_y, \nu_z$ similar

$$n_x = \frac{\nu_x}{q-q^{-1}} \qquad n_y, n_z \text{ similar}$$

Recall

$$[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}} \qquad n = 0, 1, 2, \ldots$$

## Terms

Fix integer $d \geq 0$

Let $V$ = vector space over $\mathbb{F}$ with dim $d+1$

A decomposition of $V$ is a sequence $\{V_i\}_{i=0}^{d}$

of 1-dim'l subspaces of $V$ such that

$$V = \sum_{i=0}^{d} V_i \quad \text{(dir sum)}$$

It is understood $V_{-1} = 0$, $V_{d+1} = 0$.

Given decomp $\{V_i\}_{i=0}^{d}$ of $V$

Given $\phi \in \mathrm{End}(V)$

$\phi$ is called lowering for $\{V_i\}_{i=0}^{d}$ whenever $\phi V_i \subseteq V_{i-1}$ for $0 \leq i \leq d$

... quasi lowering ... $\phi V_i \subseteq V_i + V_{i-1}$ ...

$\phi$ is called raising (resp. quasi raising) for $\{V_i\}_{i=0}^{d}$

whenever $\phi$ is lowering (resp quasi-lowering) for the

inversion $\{V_{d-i}\}_{i=0}^{d}$

$\phi$ is called multiplicity-free whenever $\phi$ is diagonalizable,

and each eigenspace of $\phi$ has dim 1

We recall the f.d. irred $U_q$-modules.

LEM 1 $\exists$ family of f.d. irreducible $U_q$-modules

$$V_{d,\varepsilon} \qquad \varepsilon \in \{1,-1\} \qquad d = 0,1,2,\ldots \qquad (*)$$

with the following property. $V_{d,\varepsilon}$ has a basis $\{v_i\}_{i=0}^{d}$ s.t.

$$k v_i = \varepsilon q^{d-2i} v_i \qquad (0 \le i \le d)$$

$$f v_i = [i+1]_q v_{i+1} \qquad (0 \le i \le d-1), \qquad f v_d = 0$$

$$e v_i = \varepsilon [d-i+1]_q v_{i-1} \qquad (1 \le i \le d), \qquad e v_0 = 0$$

Each f.d. irred $U_q$-module is iso to exactly one of the modules $*$

pf (ex)

Note For char$(\mathbb{F}) = 2$ we interp $\{1,-1\}$ to have a single element.

Referring to LEM1

call $d$ the <u>diameter</u> of $V_{d,\varepsilon}$

call $\varepsilon$ the <u>type</u> ..

Abbrev $V_d = V_{d,1}$

Note Casimir element $\Lambda$ acts on $V_{d,\varepsilon}$ as

$$\varepsilon\left(q^{d+1} + q^{-d-1}\right) I$$

**LEM 2** The $U_q$-module $V_{d,\varepsilon}$ has a basis $\{u_i\}_{i=0}^d$ on which $x, y, z$ act as follows:

$$x: \quad \varepsilon \begin{pmatrix} q^{-d} & q^{d}-q^{-d} & & & 0 \\ & q^{2-d} & q^{d}-q^{2-d} & & \\ & & \ddots & \ddots & \\ & 0 & & \ddots & \\ & & & & q^{d} \end{pmatrix} \qquad \text{"CRS } q^{d}\text{"}$$

$$y: \quad \varepsilon \operatorname{diag}(q^{d}, q^{d-2}, \ldots, q^{-d})$$

$$z: \quad \varepsilon \begin{pmatrix} q^{-d} & & & & 0 \\ q^{-d}-q^{2-d} & q^{2-d} & & & \\ & \ddots & \ddots & & \\ 0 & & \ddots & \ddots & \\ & & & q^{-d}-q^{d} & q^{d} \end{pmatrix} \qquad \text{"CRS } q^{-d}\text{"}$$

Pf Let the basis $\{v_i\}_{i=0}^{d}$ for $V_{d,\varepsilon}$ be as in Lem 1

Define

$$u_i = \gamma_i v_i \qquad 0 \le i \le d$$

where

$$\gamma_0 = 1$$
$$\gamma_i = -\varepsilon q^{d-i} \gamma_{i-1} \qquad 1 \le i \le d$$

Use Lem 1 and

$$y = k$$
$$z = k^{-1} + f(q - q^{-1})$$
$$x = k^{-1} - ek^{-1} q^{-1} (q - q^{-1})$$

to get result. □