LECTURE 30 MONDAY NOV 11   11/11/13

Recall assumptions: Integer $N \geq 1$, scalars in $\mathbb{F}$

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1} \qquad *$$

Assume $a_0 + \cdots + a_{i-1} \neq b_0 + \cdots + b_{i-1}$ $\qquad 1 \leq i \leq N$

$$a_i \neq b_j \quad \text{if} \quad i+j \leq N-1 \qquad 0 \leq i,j \leq N-1$$

Assume $*$ is feas

So by L139 for $0 \leq n \leq N$ the vectors

$$\tau_i \eta_{n-i} \qquad 0 \leq i \leq n$$

form a basis for V_n

For $1 \leq n \leq N$ we have

$$\psi : V_n \to V_{n-1}$$

$$A : V_{n-1} \to V_n$$

Also incl map

$$\varepsilon : V_{n-1} \to V_n$$

Next goal: describe how these maps act on given bases

LEM 140 For $i, j \geq 0$ s.t. $i+j \leq N-1$,

(i) $$\mathcal{E}(\tau_i z_j) = \frac{\tau_{i+1} z_j}{b_j - a_i} + \frac{\tau_i z_{j+1}}{a_i - b_j}$$

(ii) $$A(\tau_i z_j) = \frac{b_j}{b_j - a_i} \tau_{i+1} z_j + \frac{a_i}{a_i - b_j} \tau_i z_{j+1}.$$

pf (i) Use

$$1 = \frac{\lambda - a_i}{b_j - a_i} + \frac{\lambda - b_j}{a_i - b_j}$$

(ii) Use

$$\lambda = b_j \frac{\lambda - a_i}{b_j - a_i} + a_i \frac{\lambda - b_j}{a_i - b_j}$$

$\square$

Recall

$$\psi(\tau_i) = \vartheta_i \, \tau_{i-1} \qquad 0 \le i \le N$$

$$\psi(\eta_i) = \vartheta_i \, \eta_{i-1}$$

$\tau_{-1} = 0, \quad \eta_{-1} = 0$

LEM 141 For $i, j \ge 1$ s.t. $i+j \le N$

$$\psi(\tau_i \eta_j) =$$

term	coef
$\tau_{i-1}\eta_j$	$\dfrac{\eta_j(a_i)}{\eta_j(a_{i-1})}\,\vartheta_i$
$\tau_i \eta_{j-1}$	$\dfrac{\tau_i(b_j)}{\tau_i(b_{j-1})}\,\vartheta_j$

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Pf obs

$$\tau_i\eta_j \in \operatorname{Span}\{\tau_i\eta_j,\ \tau_{i+1}\eta_{j-1},\ \ldots,\ \tau_{i+j}\eta_0\}$$

$$= \operatorname{Span}\{\tau_i,\ \tau_{i+1},\ \ldots,\ \tau_{i+j}\}$$

coef of τ_i is $\eta_j(a_i)$

Apply ψ

$$\psi(\tau_i\eta_j) \in \operatorname{Span}\{\tau_{i-1},\ \tau_i,\ \ldots,\ \tau_{i+j-1}\}$$

$$= \operatorname{Span}\{\tau_{i-1}\eta_j,\ \tau_i\eta_{j-1},\ \ldots,\ \tau_{i+j-1}\eta_0\} \qquad (1)$$

coef of $\tau_{i-1}\eta_j$ is $\dfrac{\eta_j(a_i)\,\theta_i}{\eta_j(a_{i-1})}$

Interchanging roles of τ_i, η_j similarly find

$$\psi(\tau_i\eta_j) \in \operatorname{Span}\{\tau_i\eta_{j-1},\ \tau_{i-1}\eta_j,\ \ldots,\ \tau_0\eta_{i+j-1}\} \qquad (2)$$

coef of $\tau_i\eta_{j-1}$ is $\dfrac{\tau_i(b_j)\,\theta_j}{\tau_i(b_{j-1})}$

By L179

$\tau_0\eta_{i+j-1},\ \tau_1\eta_{i+j-2},\ \ldots,\ \tau_{i+j-1}\eta_0$

are lin indep. Comparing (1), (2) get results $\square$

Now assume $a_i = aq^i + a^{-1}q^{-i}$ $\quad 0 \le i \le N-1$

$b_i = bq^i + b^{-1}q^{-i}$

LEM 14.2 For $i,j \ge 1$ s.t. $i+j \le N$,

$$M^{-1}(\tau_i \eta_j) =$$

term	coef
$\tau_{i-1}\eta_{j+1}$	$\dfrac{q^j(q^i-1)(1-a^{-1}bq^{-i})}{(1-a^{-1}bq^{j-i})(1-a^{-1}bq^{j-i+1})}$
$\tau_i\eta_j$	$\dfrac{(1-q^{i+1})(1-q^{j+1}) + q(1+q^{i+j}) - ab^{-1}q^{i+1} - a^{-1}bq^{j+1}}{(1-a^{-1}bq^{j-i+1})(1-ab^{-1}q^{i-j+1})}$
$\tau_{i+1}\eta_{j-1}$	$\dfrac{q^i(q^j-1)(1-ab^{-1}q^{-j})}{(1-ab^{-1}q^{i-j})(1-ab^{-1}q^{i-j+1})}$

For $1 \le i \le N$

$M^{-1}(r_i) =$

term	coef
$r_{i-1}\, y_1$	$\dfrac{q^i - 1}{1 - a^{-1} b\, q^{1-i}}$
r_i	$q^{i-1}\, \dfrac{bq - a}{b - a q^{i-1}}$

For $1 \le j \le N$

$M^{-1}(z_j) =$

term	coef
z_j	$q^{j-1} \dfrac{aq - b}{a - bq^{j-1}}$
$\tau_1 z_{j-1}$	$\dfrac{q^j - 1}{1 - ab^{-1}q^{1-j}}$

pf First assume $i+j \le N-1$. Then use L127 along with L140, L141.

Next assume $i+j = N$.
Define $a_N = aq^N + a^{-1}q^{-N}$, $b_N = bq^N + b^{-1}q^{-N}$
and apply L127 to $\{a_i\}_{i=0}^N$, $\{b_i\}_{i=0}^N$ $\square$

Next goal: Apply DL problem to Leonard pairs.

Given LS

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

on V. Assume $d \geq 1$

Recall polys

$$\tau_i = (\lambda - \theta_0)(\lambda - \theta_1) \cdots (\lambda - \theta_{i-1}) \qquad 0 \leq i \leq d$$

$$\eta_i = (\lambda - \theta_d)(\lambda - \theta_{d-1}) \cdots (\lambda - \theta_{d-i+1})$$

and also τ_i^*, η_i^*

Recall Φ-split decomp of V:

$$\{U_i\}_{i=0}^{d}$$

$$U_0 = E_0^* V$$

$$\dim U_i = 1 \qquad 0 \leq i \leq d$$

L 143 Fa $0 \le i \le d$

(i) $(A - \theta_i I) U_i = U_{i+1}$

(ii) $(A - \theta_{d-i} I) U_i^{\Downarrow} = U_{i+1}^{\Downarrow}$

(iii) $(A^* - \theta_i^* I) U_i = U_{i+1}$

(iv) $(A^* - \theta_i^* I) U_i^{\Downarrow} = U_{i+1}^{\Downarrow}$

pf (i) $\subseteq$: By Th 67.

Suppose $\exists\, i\ (0 \le i \le d)$ s.t

$$(A - \theta_i I) U_i \ne U_{i+1}$$

Then $i \le d-1$ and

$$(A - \theta_i I) U_i = 0$$

Now

$$U_0 + U_1 + \cdots + U_i$$

is nonzero, proper subspace of V that is inv under

A and A^*. cont.

(ii)–(iv) Apply (i) to a relative of Φ □

Def 144 For $1 \le i \le d$,

by L143 U_i is invar under

$$(A - \theta_{i-1} I)(A^* - \theta_i^* I)$$

and corresp eigval is non 0. Call this eigvalue φ_i

Call $\{\varphi_i\}_{i=1}^d$ the <u>1st split sequence</u> of Φ.

Let $\{\phi_i\}_{i=1}^d$ denote the 1st split seq for $\Phi^\Downarrow$.

Call $\{\phi_i\}_{i=1}^d$ the <u>2nd split sequence</u> for Φ.

By the <u>parameter</u> <u>array</u> of Φ we mean the sequence

$$\left(\{\theta_i\}_{i=0}^d ; \{\theta_i^*\}_{i=0}^d ; \{\varphi_i\}_{i=1}^d ; \{\phi_i\}_{i=1}^d \right)$$

LEM 145 For $0 \le i \le d$

(i) $U_i = \tau_i(A) E_0^* V$

(ii) $U_i^{\Downarrow} = \eta_i(A) E_0^* V$

pf By L143 (i), (ii) $\square$

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Pick $0 \ne v \in E_0^* V$

For $0 \le i \le d$, by L145

$\tau_i(A) v$ is a basis for U_i

So $\{\tau_i(A) v\}_{i=0}^{d}$ is basis for V

rel this basis

$$A: \begin{pmatrix} \theta_0 & & & 0 \\ 1 & \theta_1 & & \\ & 1 & \ddots & \\ 0 & & \ddots & \\ & & 1 & \theta_d \end{pmatrix}$$

Lower Bidiag

$$A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & 0 \\ & \theta_1^* & \varphi_2 & \\ & & \ddots & \ddots \\ 0 & & & \varphi_d \\ & & & \theta_d^* \end{pmatrix}$$

upper Bidiag

For $0 \le i \le d$

$\tau_i(A)v$ is a basis for $U_i^{\Downarrow}$

So

$\{\tau_i(A)v\}_{i=0}^{d}$ is a basis for V.

Relative the basis

$$A: \begin{pmatrix} \theta_d & & & & 0 \\ 1 & \theta_{d-1} & & & \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ 0 & & & 1 & \theta_0 \end{pmatrix}$$

$$A^*: \begin{pmatrix} \theta_0^* & \varphi_1 & & & 0 \\ & \theta_1^* & \varphi_2 & & \\ & & \ddots & \ddots & \\ & & & & \varphi_d \\ 0 & & & & \theta_d^* \end{pmatrix}$$

Cor 14.6 The LS Φ is determined up to iso by its parameter array.

pf The above matrix entries are given by the PA. $\square$

For our LS Φ,

by th 90 $\exists\ \beta, \gamma, \gamma^*, \delta, \delta^* \in \mathbb{F}$ s.t.

$\{\theta_i\}_{i=0}^d$ is (β,γ)-rec and (β,γ,δ)-rec

$\{\theta_i^*\}_{i=0}^d$ is (β,γ^*)-rec and $(\beta,\gamma^*,\delta^*)$-rec

Assume $\beta \neq \pm 2$

By th 86 $\exists\ q,\ \alpha_i, \alpha_i^* \in \overline{\mathbb{F}}$ $(i=0,1,2)$

s.t.

$$\theta_i = \alpha_0 + \alpha_1 q^i + \alpha_2 q^{-i} \qquad 0 \le i \le d$$

$$\theta_i^* = \alpha_0^* + \alpha_1^* q^i + \alpha_2^* q^{-i}$$

$$\beta = q + q^{-1}$$

Φ is called <u>q-Racah type</u> whenever

$\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*$ are nonzero

Assume Φ is q-Racah type.

Applying an affine trans

$$A \to sA + tI, \qquad s \neq 0,\ s^* \neq 0$$

$$A^* \to s^*A^* + t^*I$$

and replacing $q \to q^2$

WLOG

$$\theta_i = a q^{2i-d} + a^{-1} q^{d-2i} \qquad 0 \le i \le d \qquad (1)$$

$$\theta^*_i = b q^{2i-d} + b^{-1} q^{d-2i} \qquad 0 \le i \le d \qquad (2)$$

Until further notice assume Φ has q-Racah type, and satisfies (1), (2).

Find φ_i, ϕ_i

Recall

$\{\theta_i\}_{i=0}^d$ are mut dist

$\{\theta^*_i\}_{i=0}^d$ are mut dist

So

- $q^{2i} \neq 1 \qquad 1 \le i \le d$
- a^2, b^2 not among $q^{2d-2}, q^{2d-4}, \ldots, q^{2-2d}$

Define

$$\vartheta_i = \sum_{h=0}^{i-1} \frac{\theta_h - \theta_{d-h}}{\theta_0 - \theta_d} \qquad 0 \le i \le d+1$$

So

$$\gamma_0 = 0, \quad \gamma_1 = 1, \quad \gamma_d = 1, \quad \gamma_{d+1} = 0$$

$$\gamma_i = \gamma_{d-i+1} \qquad 0 \le i \le d+1$$

One checks

$$\gamma_i = \frac{q^i - q^{-i}}{q - q^{-1}} \quad \frac{q^{d-i+1} - q^{i-d-1}}{q^d - q^{-d}} \qquad 0 \le i \le d+1$$

So

$$\gamma_i \ne 0 \qquad 1 \le i \le d$$

thm 147 (Sarah Bockting-Conrad) With above notation,

$\exists \quad \psi \in \mathrm{End}(V)$ that sends

$$\tau_i(A)v \longrightarrow \vartheta_i \, \tau_{i-1}(A)v \qquad 0 \le i \le d$$

$$\eta_i(A)v \longrightarrow \vartheta_i \, \eta_{i-1}(A)v$$

where $0 \ne v \in E_0^* V$

pf the data

$a_i = \theta_i, \quad b_i = \theta_{d-i} \qquad 0 \le i \le d-1$

is feasible in the sense of the DL problem

Define $V' = \mathrm{Span}(1, \lambda, \dots, \lambda^d)$

$\exists$ iso of $\mathbb{F}$-vector spaces

$$V' \longrightarrow V$$
$$f \mapsto f(A)v$$

$\exists \quad \psi' \in \mathrm{End}(V')$ that sends

$$\tau_i \to \vartheta_i \, \tau_{i-1} \qquad 0 \le i \le d.$$

$$\eta_i \to \vartheta_i \, \eta_{i-1}$$

Transport ψ' to $\mathrm{End}(V)$ via above iso $V' \to V$ $\square$