Lecture 2 Friday Sept 6

Chapter I

TD pairs in graph theory

For this chapter $\mathbb{F} = \mathbb{C}$

let X = nonempty finite set

$n = |X|$

$\mathrm{Mat}_X(\mathbb{C})$ = $\mathbb{C}$-algebra of all $n \times n$ matrices with entries in $\mathbb{C}$

View rows/cols as indexed by X

$\mathbb{C}^X$ = vector space $\mathbb{C}^n$ (column vectors) with coords indexed by X

$\mathrm{Mat}_X(\mathbb{C})$ acts on $\mathbb{C}^X$ by left mult

Abbrev

$V = \mathbb{C}^X$ "standard module"

Endow V with Hermitian dot product

$$\langle u, v \rangle = u^t \bar{v} \qquad u, v \in V$$

Obs

$$\langle Bu, v \rangle = \langle u, \bar{B}^t v \rangle \qquad \forall\, B \in Mat_X(\mathbb{C})$$

For $x \in X$ define $\hat{x} \in V$ by

$$\hat{x} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow x\text{-coord}$$

So

$\{ \hat{x} \mid x \in X \}$ is ortho-normal basis for V

A matrix $B \in Mat_X(\mathbb{C})$ is <u>Hermitian</u> whenever

$$\bar{B}^t = B$$

In this case B is diagonalizable and its eigenvalues are in $\mathbb{R}$ (ex)

A *graph* is a pair $\Gamma = (X, R)$ where

X = nonempty finite set (the *vertices*)

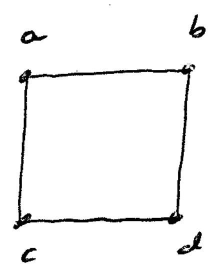
R = set of distinct 2-element subsets of X (the *edges*)

Vertices $x, y \in X$ are called *adjacent* whenever $xy \in R$

[Note our graphs are undirected, without loops or multiple edges]

Ex

Γ:



"4-cycle"

$X = \{a, b, c, d\}$ $\quad$ $R = \{ab, bd, ac, cd\}$

For a graph $\Gamma = (X, R)$ the <u>adjacency matrix</u>

$A \in Mat_X(\mathbb{C})$ satisfies

$$A_{xy} = \begin{cases} 1 & \text{if } xy \in R \\ 0 & \text{if } xy \notin R \end{cases} \qquad x, y \in X$$

Ex (4-cycle)

$$A: \quad \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 1 & 1 & 0 \\ b & 1 & 0 & 0 & 1 \\ c & 1 & 0 & 0 & 1 \\ d & 0 & 1 & 1 & 0 \end{array}$$

Note that

$$\bar{A} = A, \qquad A^t = A$$

so A is Hermitean

let M = subalgebra of $Mat_X(\mathbb{C})$ gen by A

"adjacency algebra"

By an <u>eigenvalue of Γ</u> we mean an eigenvalue of A.

For an eigenvalue θ of Γ,
by the <u>multiplicity</u> of θ we mean the dimension of the corresp eigenspace.

Ex (4-cycle)

eigenvalue	2	0	-2
mult	1	2	1

For $x \in X$ define

$$\Gamma(x) = \{ y \in X \mid xy \in R \}$$

Obs

$$A\hat{x} = \sum_{y \in \Gamma(x)} \hat{y}$$

Given $x, y \in X$

Given integer $l \geq 0$

A <u>walk of length l from x to y</u> is a sequence of vertices

$$x_0, x_1, \ldots, x_l$$

such that

$$x_0 = x, \qquad x_l = y$$

$$x_{i-1}, x_i \text{ adj for } 1 \leq i \leq l$$

Exercise

$$(A^l)_{xy} = \text{number of walks of length } l \text{ from } x \text{ to } y$$

For $x \in X$ define

$$k(x) = |\Gamma(x)| \qquad \text{"valency of } x\text{"}$$

Call Γ <u>regular with valency k</u> whenever

$$k(x) = k \qquad \forall\, x \in X$$

Ex 4-cycle is regular with valency $k = 2$

Ex Let θ_{max} = max'l eigenvalue of Γ

(i) $\theta_{max} \leq \max\{k(x) \mid x \in X\}$

(ii) Suppose Γ is regular with valency k.

then $\theta_{max} = k$

Γ is <u>connected</u> whenever $\forall x, y \in X$ $\exists$ walk from x to y.

Assume Γ is connected. For $x, y \in X$ the <u>distance</u>

$$\partial(x,y) = \min\{\ell \mid \exists \text{ walk of length } \ell \text{ from } x \text{ to } y\}$$

For $x \in X$ define

$$D_x = \max\{\partial(x,y) \mid y \in X\}$$ "diameter wrt x"

$$D = \max\{D_x \mid x \in X\}$$ "diameter of Γ"

Ex (4-cycle)

$D_x = 2 \quad \forall x \in X,$

$D = 2.$

Let $\{\theta_i\}_{i=0}^{\mathcal{D}}$ denote an ordering of the eigenvalues of Γ.

For $0 \le i \le \mathcal{D}$ let E_i denote the prim idempotent of A for θ_i

So

$$E_i V = \text{eigenspace of } A \text{ for } \theta_i$$

Call $\mathcal{D}$ the <u>dual</u> <u>diameter</u> of Γ.

For notational convenience define $E_i = 0$ for $i < 0$ or $i > \mathcal{D}$

Obs
$$V = E_0 V + E_1 V + \cdots + E_{\mathcal{D}} V \quad \text{(orthog dir sum)}$$

Ex A connected graph with diameter D has at least $D+1$ distinct eigenvalues.

In other words

$$\mathcal{D} \ge D.$$

Ex (4-cycle) $\mathcal{D} = 2$

Write
$$\theta_0 = 2 \qquad \theta_1 = 0 \qquad \theta_2 = -2$$

then

$$E_0 = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$E_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$E_2 = \frac{1}{4}\begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Until further notice, assume Γ is connected and fix $x \in X$ "the base vertex"

For $0 \le i \le D_x$ define a diagonal matrix $E_i^* = E_i^*(x)$ in $\mathrm{Mat}_X(\mathbb{C})$ by

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } \partial(x,y) = i \\ 0 & \text{if } \partial(x,y) \neq i \end{cases} \qquad y \in X$$

Call E_i^* the <u>ith dual idempotent of Γ wrt x</u>

For convenience define
$$E_i^* = 0 \quad \text{if } i < 0 \text{ or } i > D_x$$

Obs:
$$E_i^* E_j^* = \delta_{ij} E_i^* \qquad (0 \le i, j \le D_x)$$
$$I = \sum_{i=0}^{D_x} E_i^*$$

$\{E_i^*\}_{i=0}^{D_x}$ form a basis for a commutative subalgebra of $\mathrm{Mat}_X(\mathbb{C})$, denoted $M^* = M^*(x)$

Call this the <u>dual adjacency algebra of Γ wrt x</u>

Ex (4-cycle)

$$E_0^* = \mathrm{diag}(1\ 0\ 0\ 0)$$

$$E_1^* = \mathrm{diag}(0\ 1\ 1\ 0)$$

$$E_2^* = \mathrm{diag}(0\ 0\ 0\ 1)$$

For $0 \le i \le D_x$

$$E_i^* V = \mathrm{Span}\{\hat{y} \mid y \in X,\ \partial(x,y) = i\}$$

"ith subconstituent of Γ wrt x"

obs

$$V = E_0^* V + E_1^* V + \cdots + E_{D_x}^* V \qquad (\text{ods})$$

By the triangle inequality

$$A E_i^* V \subseteq E_{i-1}^* V + E_i^* V + E_{i+1}^* V \qquad (0 \le i \le D_x) \quad ☆$$

In other words

$$E_i^* A E_j^* = 0 \text{ if } |i-j| > 1 \qquad (0 \le i,j \le D_x)$$

Def 1 Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by M and M^*.

Call T the <u>subconstituent algebra of Γ wrt x</u>

Note that T is generated by A and $\{E_i^*\}_{i=0}^{D_x}$

Ex (4 cycle)

Find a basis for T

Find a basis for the center of T

The algebra T is closed under complex conjugation and transpose. Therefore T is semi simple.

We dont need the full Wedderburn theory, only the following facts.

By a <u>T-module</u> we mean a subspace $W \subseteq V$ s.t. $BW \subseteq W$ for all $B \in T$.

Let W denote a T-module.

W is called <u>irreducible</u> whenever $W \neq 0$ and W does not contain a T-module other than 0, W.

LEM 2 Let U denote a T-module and let W denote a T-module contained in U. Then there exists a T-module W' such that

$$U = W + W' \qquad (\text{ods}) \qquad (*)$$

pf Define

$$W' = \{ u \in U \mid \langle u, w \rangle = 0 \ \forall w \in W \}$$

Then $(*)$ holds by elem. lin algebra.

Show W' is a T-module

Pick $u \in W'$ and $B \in T$. Show $Bu \in W'$.

To do this, pick $w \in W$ and show $\langle Bu, w \rangle = 0$.

Recall $\overline{B}^t \in T$ so $\overline{B}^t w \in W$. So

$$\langle Bu, w \rangle = \langle u, \overline{B}^t w \rangle$$
$$= 0.$$

$\square$

COR 3 Any T-module is an orthog direct sum of irred T-modules.
In particular the standard module V is an orthog direct sum of irred T-modules.

pf By L2. $\square$