LECTURE 29 FRIDAY NOV 8

11/8/13

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[No class MWF Nov 18–22. I am at a conference in China.]

The following result is inspired by Pr 136

Prop 137 Given integer $N \geq 2$ and scalars in $\mathbb{F}$: *

$$\{a_i\}_{i=0}^{N} \qquad \{b_i\}_{i=0}^{N}$$

Assume $a_1 \neq b_0$, $a_1 \neq b_1$

Then TFAE

(i) For $0 \leq i \leq j \leq N$

$$(a_i - b_j)(a_0 + \cdots + a_{j-i} - b_0 - \cdots - b_{j-i})$$

$$= (a_0 - b_{j-i})(a_i + \cdots + a_j - b_i - \cdots - b_j)$$

(ii) $\exists\ \beta, \gamma, \delta \in \mathbb{F}$ s.t each of * is

(β, γ)-rec and (β, γ, δ)-rec.

pf (sketch)

(ii) → (i) Write each sequence x in closed form

(i) → (ii) Using $i=1, j=2$ get

$$(a_0-b_1)(b_1-a_2) = (b_0-a_1)(a_1-b_2)$$

Now as in pf of Th 136, $\exists\ \beta, \gamma, \delta \in \mathbb{F}$ s.t each of $\{a_i\}_{i=0}^{2}$, $\{b_i\}_{i=0}^{2}$ is (β, γ)-rec and (β, γ, δ)-rec

Let $3 \le j \le N$. For $i=j-1$ get

$$(a_{j-1}-b_j)(a_0+a_1-b_0-b_1) \tag{1}$$
$$= (a_0-b_1)(a_{j-1}+a_j-b_{j-1}-b_j)$$

For $i=j-2$ get

$$(a_{j-2}-b_j)(a_0+a_1+a_2-b_0-b_1-b_2) \tag{2}$$
$$= (a_0-b_2)(a_{j-2}+a_{j-1}+a_j-b_{j-2}-b_{j-1}-b_j)$$

View (1), (2) as linear eqs

in unknowns a_j, b_j

One checks coef matrix has det

$$(a_1-b_0)(a_1-b_1) \neq 0$$

System has unique solution

$$a_j = \frac{(a_1+a_2-b_0-b_1)L_j + (b_0-a_1)R_j}{(a_1-b_0)(a_1-b_1)} \tag{3}$$

$$b_j = \frac{(b_2-a_0)L_j + (a_0-b_1)R_j}{(a_1-b_0)(a_1-b_1)} \tag{4}$$

where

$$L_j = a_{j-1}(a_1-b_0) - b_{j-1}(b_1-a_0)$$

$$R_j = a_{j-2}(a_0+a_1+a_2-b_0-b_1-b_2)$$
$$- (a_0-b_2)(a_{j-2}+a_{j-1}-b_{j-2}-b_{j-1})$$

By ind on j each of

$$\{a_i\}_{i=0}^{j+1}, \qquad \{b_i\}_{i=0}^{j+1}$$

is (β,γ)-rec and (β,γ,δ)-rec.

Solving for a_j, b_j using (3), (4) we find each of

$$\{a_i\}_{i=0}^{j} \qquad \{b_i\}_{i=0}^{j}$$

is (β,γ)-rec and (β,γ,δ)-rec.

Result follows. $\square$

In Prop 137 consider the special case

$$b_i = a_{N-i} \qquad 0 \le i \le N$$

Prop 138 Given mutually distinct scalars in $\mathbb{F}$:

$$\{a_i\}_{i=0}^{N}$$

TFAE

(i) For $0 \le i \le j \le N$

$$(a_i - a_{N-j})(a_0 + \cdots + a_{j-i} - a_N - \cdots - a_{N-j+i})$$

$$= (a_0 - a_{N-j+i})(a_i + \cdots + a_j - a_{N-i} - \cdots - a_{N-j})$$

(ii) the scalar

$$\frac{a_{i-2} - a_{i+1}}{a_{i-1} - a_i}$$

is indep of i for $2 \le i \le N-1$ $\square$

pf Apply Prop 137 with $b_i = a_{N-i}$ for $0 \le i \le N$

Back to the general DL problem.

Given integer $N \geq 1$ and scalars in $\mathbb{F}$,

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1} \qquad *$$

assume

$$a_0 + \cdots + a_{i-1} \neq b_0 + \cdots + b_{i-1} \qquad 1 \leq i \leq N$$

Assume $*$ feas.

Consider these polynomials:

$$1$$

$$\lambda - b_0 \quad — \quad \lambda - a_0$$

$$(\lambda - b_0)(\lambda - b_1) \quad — \quad (\lambda - a_0)(\lambda - b_0) \quad — \quad (\lambda - a_0)(\lambda - a_1)$$

$$(\lambda - b_0)(\lambda - b_1)(\lambda - b_2) \quad — \quad (\lambda - a_0)(\lambda - b_0)(\lambda - b_1) \quad — \quad (\lambda - a_0)(\lambda - a_1)(\lambda - b_0) \quad — \quad (\lambda - a_0)(\lambda - a_1)(\lambda - a_2)$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Fix $0 \le i \le n \le N$

ith poly in row n is $\tau_i \eta_{n-i}$

LEM 13.9 Fix $0 \le n \le N$ TFAE

(i) the polynomials

$$\tau_i \eta_{n-i} \qquad 0 \le i \le n$$

are lin indep

(ii) $a_r \ne b_s$ if $r+s \le n-1$ $(0 \le r, s \le N-1)$

pf (i) → (ii) Suppose $\exists$ $r, s \ge 0$ s.t $r+s \le n-1$ and $a_r = b_s$.

Then

$$\lambda - a_r = \lambda - b_s$$

is a factor in $\tau_i \eta_{n-i}$ for $0 \le i \le n$

So $\{\tau_i \eta_{n-i}\}_{i=0}^{n}$ dont span V_n. They must be lin dependent.

(ii) $\to$ (i) For $0 \le j \le n$ write

$\tau_j \eta_{n,j}$ in the basis

$1, \lambda, \lambda^2, \ldots, \lambda^n$

Put coefs in col j of a matrix $C \in Mat_{n+1}(\mathbb{F})$

So for $n=2$

$$C = \begin{array}{cc} \begin{bmatrix} 1 & 1 & 1 \\ -b_0-b_1 & -a_0-b_0 & -a_0-a_1 \\ b_0b_1 & a_0b_0 & a_0a_1 \end{bmatrix} & \begin{array}{c} \lambda^2 \\ \lambda \\ 1 \end{array} \end{array}$$

show $\det(C) \neq 0$ Find $\det(C)$ "identity factors"

View a_i, b_i as indets

For $0 \le i \le n$ each entry in row i of C is hom deg i

$\det C$ is homog poly in $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$ with total

degree $0+1+2+\cdots+n = \binom{n+1}{2}$

Obs $\det(C)$ has the following factors

$$a_i - b_j \qquad 0 \le i, j \qquad i+j \le n-1 \qquad *$$

List $*$ has $\binom{n+1}{2}$ terms

$\exists\ \alpha \in \mathbb{F}$ s.t.

$$\det(C) = \alpha \prod_{\substack{0 \le i, j \\ i+j \le n-1}} (a_i - b_j)$$

To find α set

$$a_i = 1 \qquad b_i = 0 \qquad 0 \le i \le n-1$$

So

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \\ & -1 & -2 & -3 & \cdots \\ & & 1 & 3 & \cdots \\ & 0 & & -1 & \\ & & & & \ddots \end{pmatrix}$$

$$\alpha = \det(C) = \pm 1$$

$$\alpha \neq 0$$

We now see $\det C \neq 0$

Result follows. $\square$