Recall the data

$a_i = aq^i + a^{-1}q^{-i}$ $\quad 0 \le i \le N-1$

$b_i = bq^i + b^{-1}q^{-i}$

$a \ne b$

$q^i \ne 1 \qquad abq^{i-1} \ne 1 \qquad 1 \le i \le N$

Data is feasible.

Apply any affine trans

$a_i \to s a_i + t$ $\quad s, t \in \mathbb{F} \quad s \ne 0$

$b_i \to s b_i + t$

(still feasible)

Resulting sequences satisfy:

$\exists\ \beta, \gamma, \delta \in \mathbb{F}$ s.t

each sequence is (β,γ)-rec, (β,γ,δ)-rec

Good guess: Given $\beta, \gamma, \delta \in \mathbb{F}$

Given sequences

$\{a_i\}_{i=0}^{N-1}, \quad \{b_i\}_{i=0}^{N-1}$ $\quad *$

in $\mathbb{F}$ that are both (β,γ)-rec and (β,γ,δ)-rec

Assume $a_0 + \cdots + a_{i-1} \ne b_0 + \cdots + b_{i-1}$ $\quad 1 \le i \le N-1$.

then $*$ is feas. Guess is correct, as we will see.

Define

$$a_i = a q^{-i} \qquad b_i = q^{-i} \qquad 0 \le i \le N-1$$

$$q^i \ne 1 \qquad 1 \le i \le N \qquad a \ne 0, 1$$

Here

$$\vartheta_i = \frac{1-q^i}{1-q} q^{1-i} = \frac{1-q^{-i}}{1-q^{-1}} \qquad 0 \le i \le N$$

For $0 \le i \le j \le N$

$$\begin{bmatrix} j \\ i \end{bmatrix}_\vartheta = \frac{(-1)^i (q^{-j};q)_i \, q^i \, q^{\binom{i}{2}}}{(q;q)_i}$$

For $0 \le i \le N$

$$\tau_i = (-1)^i a^i q^{-\binom{i}{2}} (a^{-1}\lambda; q)_i$$

$$\eta_i = (-1)^i q^{-\binom{i}{2}} (\lambda; q)_i$$

We have $\mathcal{L} \ne 0$. The proof uses the identity

$$\frac{(\lambda;q)_j}{(a;q)_j} = {}_2\varphi_1\left(\begin{matrix} q^{-j} \quad \lambda a^{-1} \\ a^{-1} q^{1-j} \end{matrix} \,\Big|\, q; q \right) \qquad 0 \le j \le N$$

Here

$$\Delta = \exp_{q^{1/2}}(a\psi)\ \exp_{q^{-1/2}}(-\psi)$$

$$= \sum_{i=0}^{N} \frac{(a;q)_i (q-1)^i}{(q;q)_i} \psi^i$$

the proof uses q-Binom thm

$$(a;q)_j = {}_1\varphi_0\left(\begin{matrix} q^{-j} \\ - \end{matrix} \Big/ q, q^j a \right)$$

— o —

Define

$$a_i = aq^i \qquad b_i = q^{-i} \qquad 0 \le i \le N-1$$

$$q^i \ne 1 \qquad aq^{i-1} \ne 1 \qquad 1 \le i \le N, \qquad a \ne 0$$

Here

$$\theta_i = \frac{1-q^i}{1-q}\ \frac{1-aq^{i-1}}{1-a}\ q^{1-i} \qquad 0 \le i \le N$$

For $0 \le i \le j \le N$

$$\left[\begin{matrix} j \\ i \end{matrix} \right]_\theta = \frac{(q^{-j};q)_i\,(a^{-1}q^{1-j};q)_i\ q^{ij}\,a^i}{(q;q)_i\ (a;q)_i}$$

For $0 \le i \le N$

$$\tau_i = \lambda^i (a\lambda^{-1}; q)_i$$

$$\eta_i = (-1)^i q^{-\binom{i}{2}} (\lambda; q)_i$$

We have $\mathcal{L} \neq 0$. the proof uses

$$\frac{(\lambda; q)_j}{(a; q)_j} = {}_2\varphi_1 \left(\begin{matrix} q^{-j} & a\lambda^{-1} \\ & a \end{matrix} \middle| \, q, \; \lambda q^j \right)$$

$0 \le j \le N$

Here

$$\Delta = \exp_{q^{-1/2}} \left((a-1) \psi \right)$$

— o —

there are feasible data sequences of a similar nature, with $\beta = 2$ $(q=1)$ and $\beta = -2$ $(q=-1)$ not listed here.

— o —

Next goal

Given integer $N \geq 1$ and scalars in $\mathbb{F}$

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1} \qquad *$$

s.t.

$$a_0 + \cdots + a_{i-1} \neq b_0 + \cdots + b_{i-1} \qquad 1 \leq i \leq N$$

Assume $*$ is feasible.

Given $a_N, b_N \in \mathbb{F}$. Find nec/suf conds on a_N, b_N

s.t.

$$\{a_i\}_{i=0}^{N}, \qquad \{b_i\}_{i=0}^{N} \qquad **$$

is feas.

Recall

$$\tau_{N+1} = (\lambda - a_N)\,\tau_N$$

$$\eta_{N+1} = (\lambda - b_N)\,\eta_N$$

$$g_{N+1} = g_N + \frac{a_N - b_N}{a_0 - b_0}$$

Assume $**$ is feas. then

$$\psi\,\tau_{N+1} = g_{N+1}\,\tau_N$$

$$\psi\,\eta_{N+1} = g_{N+1}\,\eta_N$$

So
$$\psi(\eta_{N+1} - \tau_{N+1}) = g_{N+1}(\eta_N - \tau_N)$$

$$\eta_{N+1} - \tau_{N+1} = (\lambda - b_N)\eta_N - (\lambda - a_N)\tau_N$$
$$= \lambda(\eta_N - \tau_N) + a_N \tau_N - b_N \eta_N$$

So
$$g_{N+1}(\eta_N - \tau_N) = \psi(\eta_{N+1} - \tau_{N+1})$$
$$= \psi(\lambda(\eta_N - \tau_N)) + a_N \underset{g_N \tau_{N-1}}{\psi(\tau_N)} - b_N \underset{g_N \eta_{N-1}}{\psi(\eta_N)}$$

So
$$\psi(\lambda(\eta_N - \tau_N)) + a_N g_N \tau_{N-1} - b_N g_N \eta_{N-1} - g_{N+1}(\eta_N - \tau_N) \quad (\star)$$

is zero.

We now write $(\star)$ in the basis $\{\tau_i\}_{i=0}^{N}$

By thm 102

$$\eta_N = \sum_{i=0}^{N} \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_q T_i$$

So

$$\eta_N - T_N = \sum_{i=0}^{N-1} \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_q T_i$$

Find $\lambda(\eta_N - T_N)$:

For $0 \le i \le N-1$

$$\lambda T_i = a_i T_i + T_{i+1}$$

So

$$\lambda(\eta_N - T_N) = \sum_{i=0}^{N-1} a_i \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix}_q T_i$$

$$+ \sum_{i=1}^{N} \eta_{N-i+1}(a_0) \begin{bmatrix} N \\ i-1 \end{bmatrix}_q T_i$$

Find $\psi(\lambda(\eta_N - \tau_N))$

Recall $\psi(\tau_i) = \beta_i \tau_{i-1}$ $\qquad 0 \le i \le N$

So

$$\psi(\lambda(\eta_N - \tau_N)) =$$

$$\sum_{i=0}^{N-2} \beta_{i+1} a_{i+1} \eta_{N-i-1}(a_0) \begin{bmatrix} N \\ i+1 \end{bmatrix} \tau_i$$

$$+ \sum_{i=0}^{N-1} \beta_{i+1} \eta_{N-i}(a_0) \begin{bmatrix} N \\ i \end{bmatrix} \tau_i$$

Find η_{N-1}: By M102

$$\eta_{N-1} = \sum_{i=0}^{N-1} \eta_{N-i-1}(a_0) \begin{bmatrix} N-1 \\ i \end{bmatrix}_\beta \tau_i$$

In (✡) the coef of τ_i is

$$\eta_{N-i-1}(a_0)$$

times

$$(a_{i+1} - b_N)\, g_{N-i} + (b_{N-i-1} - a_0)(g_{N+1} - g_{i+1}) \qquad (✡✡)$$

for $0 \le i \le N-2$, and

$$0$$

for $i = N-1, N$.

We require coef of τ_i is 0 for $0 \le i \le N$.

To avoid trivialities assume

$N \ge 3$, $\quad a_0 \ne b_1$

then for $i = N-2, N-3$

$$\eta_{N-i-1}(a_0) \ne 0$$

So ✡✡ is 0

Gives linear 2×2 system of eqs in unknowns

$$b_N,\ g_{N+1}$$

check coef matrix is nonsingular:

Require

$$g_3 b_N + (a_0 - b_2) g_{N+1} = a_{N-2} g_3 + (a_0 - b_2) g_{N-2}$$

$$g_2 b_N + (a_0 - b_1) g_{N+1} = a_{N-1} g_2 + (a_0 - b_1) g_{N-1}$$

Coef matrix is

$$C = \begin{pmatrix} g_3 & a_0 - b_2 \\ g_2 & a_0 - b_1 \end{pmatrix}$$

Find det

$$g_3 = g_2 + \frac{a_2 - b_2}{a_0 - b_0}$$

$$\begin{pmatrix} \dfrac{a_2 - b_2}{a_0 - b_0} & b_1 - b_2 \\ \dfrac{a_0 + a_1 - b_0 - b_1}{a_0 - b_0} & a_0 - b_1 \end{pmatrix} \quad r_1' = r_1 - r_2$$

$$\begin{pmatrix} a_2-b_2 & b_1-b_2 \\ a_0+a_1-b_0-b_1 & a_0-b_1 \end{pmatrix} \qquad c_1' = c_1(a_0-b_0)$$

$$C' = \begin{pmatrix} a_2-b_1 & b_1-b_2 \\ a_1-b_0 & a_0-b_1 \end{pmatrix} \qquad c_1' = c_1 - c_2$$

By L103 $\quad (a_0-b_1)(b_1-a_2) = (b_0-a_1)(a_1-b_2)$

$$\det C' = (a_2-b_1)(a_0-b_1) - (b_1-b_2)(a_1-b_0)$$

$$= (a_1-b_0)(a_1-b_1)$$

So $\det(C) = \dfrac{(a_1-b_0)(a_1-b_1)}{a_0-b_0}$ ✓

Assume

$$a_1 \neq b_0, \quad a_1 \neq b_1$$

so C^{-1} exists and 2×2 system has unique sol

for b_{N-1}, g_{N+1} (and hence a_N)

Get "extention equations"

$$\frac{b_N}{a_0-b_0} = \frac{(a_0-b_1)\left(a_{N-2}g_3+(a_0-b_2)g_{N-2}\right)+(b_2-a_0)\left(a_{N-1}g_2+(a_0-b_1)g_{N-1}\right)}{(a_1-b_0)(a_1-b_1)}$$

$$\frac{g_{N+1}}{a_0-b_0} = \frac{g_3\left(a_{N-1}g_2+(a_0-b_1)g_{N-1}\right)-g_2\left(a_{N-2}g_3+(a_0-b_2)g_{N-2}\right)}{(a_1-b_0)(a_1-b_1)}$$

to get a_N use

$$g_{N+1} = g_N + \frac{a_N-b_N}{a_0-b_0}$$

ex Assume

$$a_i = aq^i + a^{-1}q^{-i} \qquad 0 \le i \le N-1$$

$$b_i = bq^i + b^{-1}q^{-i}$$

Get

$$a_N = aq^N + a^{-1}q^{-N}$$

$$b_N = bq^N + b^{-1}q^{-N}$$

thm 136 Given integer $N \geq 3$ and scalars in $\mathbb{F}$:

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1} \qquad *$$

s.t.

$$a_0 + \cdots + a_{i-1} \neq b_0 + \cdots + b_{i-1} \qquad 1 \leq i \leq N$$

Then $*$ is feas iff at least one of (i) – (iv) hold

(i) $a_{i-1} = b_i \qquad 1 \leq i \leq N-1$

(ii) $b_{i-1} = a_i \qquad 1 \leq i \leq N-1$

(iii) $\exists\ \theta \in \mathbb{F}$ s.t

$$a_0 \neq \theta, \quad b_0 \neq \theta$$

$$a_i = \theta \quad b_i = \theta \qquad 1 \leq i \leq N-2$$

$$\frac{\theta - a_{N-1}}{\theta - b_0} = \frac{\theta - b_{N-1}}{\theta - a_0}$$

(iv) $\exists\ \beta, \gamma, \delta \in \mathbb{F}$ s.t

$$a_{i-1} - \beta a_i + a_{i+1} = \gamma \qquad 1 \leq i \leq N-2$$
$$b_{i-1} - \beta b_i + b_{i+1} = \gamma$$

$$a_{i-1}^2 - \beta a_{i-1} a_i + a_i^2 - \gamma(a_{i-1} + a_i) = \delta \qquad 1 \leq i \leq N-1$$
$$b_{i-1}^2 - \beta b_{i-1} b_i + b_i^2 - \gamma(b_{i-1} + b_i) = \delta$$

Pf (sketch)

First assume at least one of (i)—(iv)

One checks * is feas.

[In (iv) we checked this for $\beta \neq 2, -2$, the cases
$\beta = 2$, $\beta = -2$ are sim]

Next assume * is feas. Show at least one of (i)—(iv) holds.

Cases

I $a_0 = b_1$

II $b_0 = a_1$

III $a_0 \neq b_1$, $b_0 \neq a_1$, $a_1 = b_1$

IV $a_0 \neq b_1$, $b_0 \neq a_1$, $a_1 \neq b_1$

Cases I – III routine (ex)

Assume Case IV show (iv) holds.

Use induction on N.

First assume $N = 3$

Since $*$ is feasible

$$(a_0 - b_1)(b_1 - a_2) = (b_0 - a_1)(a_1 - b_2) \qquad (1)$$

Since $a_1 \neq b_1$ $\exists$ $\beta, \gamma \in \mathbb{F}$ s.t.

$$a_0 - \beta a_1 + a_2 = \gamma,$$
$$b_0 - \beta b_1 + b_2 = \gamma$$

Use these eqs to elim a_2, b_2 in (1). This gives

$$(a_0 - b_1)(b_1 + a_0 - \beta a_1 - \gamma) = (b_0 - a_1)(a_1 + b_0 - \beta b_1 - \gamma) \qquad (2)$$

obs. $\{a_i\}_{i=0}^{2}$ is (β, γ)-rec.

So $\exists$ $\delta \in \mathbb{F}$ s.t. $\{a_i\}_{i=0}^{2}$ is (β, γ, δ)-rec

ie

$$a_{i-1}^2 - \beta a_{i-1} a_i + a_i^2 - \gamma(a_{i-1} + a_i) = \delta \qquad i = 1, 2$$

Sim $\exists$ $\delta' \in \mathbb{F}$ s.t. $\{b_i\}_{i=0}^{2}$ is (β, γ, δ')-rec

Show $\delta = \delta'$. Show

$$a_0^2 - \beta a_0 a_1 + a_1^2 - \gamma(a_0 + a_1)$$
$$= b_0^2 - \beta b_0 b_1 + b_1^2 - \gamma(b_0 + b_1)$$

This is (2) in disguise

Now assume $N \geq 4$

By construction

$$\{a_i\}_{i=0}^{N-2}, \quad \{b_i\}_{i=0}^{N-2} \tag{3}$$

is feasible and satisfies Case IV

So $\exists\ \beta, \gamma, \delta \in \mathbb{F}$ s.t. each sequence (3)

is (β,γ)-rec and (β,γ,δ)-rec

Now write (3) in closed form, and plug into

the extention equations to get a_{N-1}, b_{N-1}

one finds each of $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ is

(β,γ)-rec and (β,γ,δ)-rec. ☐