LECTURE 27 MONDAY NOV 4 11/4/13

Recall our data

$$a_i = aq^i + a^{-1}q^{-i} \qquad 0 \le i \le N-1$$

$$b_i = bq^i + b^{-1}q^{-i}$$

$$a \neq b$$

$$q^i \neq 1 \qquad abq^{i-1} \neq 1 \qquad 1 \le i \le N$$

Data is feasible

Recall vectors $\{w_i\}_{i=0}^{N-1}$ and maps

$$K, B, M$$

Next goal: Find actions of K, B, M

on τ_i, η_i, w_i

LEM 124 For $0 \le i \le N$,

(i) $$M^{-1} \tau_i = q^i \tau_i + (q-1)(a-b^{-1}) \jmath_i q^{i-1} \tau_{i-1}$$

$$= q^i \tau_i + (q^i-1)(a q^{i-1} - b^{-1}) \tau_{i-1}$$

(ii) $$M^{-1} \eta_i = q^i \eta_i + (q-1)(b-a^{-1}) \jmath_i q^{i-1} \eta_{i-1}$$

$$= q^i \eta_i + (q^i-1)(b q^{i-1} - a^{-1}) \eta_{i-1}$$

pf (i) By Prop 118,

$$M^{-1} \tau_i = K^{-1}\Big(1 + (q-1)(a-b^{-1})\psi\Big) \tau_i$$

$$= K^{-1}\Big(\tau_i + (q-1)(a-b^{-1}) \jmath_i \tau_{i-1}\Big)$$

$$= q^i \tau_i + (q-1)(a-b^{-1}) \jmath_i q^{i-1} \tau_{i-1}$$

Now eval $\jmath_i$ using L106

(ii) Sim

$\square$

LEM 125 For $0 \le i \le N$,

(i) $$K w_i = q^{-i} w_i + (q-1)(a-b^{-1}) q^{-i} \mathfrak{J}_i w_{i-1}$$

$$= q^{-i} w_i + (1-q^{-i})(a-b^{-1}q^{1-i}) w_{i-1}$$

(ii) $$B w_i = q^{-i} w_i + (q-1)(b-a^{-1}) q^{-i} \mathfrak{J}_i w_{i-1}$$

$$= q^{-i} w_i + (1-q^{-i})(b-a^{-1}q^{1-i}) w_{i-1}$$

pf (i) By Prop 118,

$$K w_i = \left(1 + (q-1)(a-b^{-1})\,\psi\right) \underbrace{M w_i}_{= q^{-i} w_i}$$

$$= q^{-i} w_i + (q-1)(a-b^{-1}) \mathfrak{J}_i w_{i-1}$$

Now eval $\mathfrak{J}_i$ using L106

(ii) Sim

$\square$

Recall

$$V = \mathrm{Span}\{\lambda^i\}_{i=0}^N \subseteq \mathbb{F}[\lambda]$$

For $0 \le n \le N$

$$V_n = \mathrm{Span}\{\tau_i\}_{i=0}^n = \mathrm{Span}\{\eta_i\}_{i=0}^n = \mathrm{Span}\{\omega_i\}_{i=0}^n$$

$$V_N = V$$

Recall

$$A : \begin{array}{ccc} V_{N-1} & \longrightarrow & V_N \\ v & \longmapsto & \lambda v \end{array}$$

Next goal: how does A relate to

K, B, M, ψ

LEM 126 On V_{N-1},

(i) $$\frac{qKA - AK}{q-1} = a^{-1}K^2 + aI$$

(ii) $$\frac{qBA - AB}{q-1} = b^{-1}B^2 + bI$$

pf (i) For $0 \le i \le N-1$ Apply each side to τ_i

$$KA\tau_i = K(a_i\tau_i + \tau_{i+1})$$
$$= a_i q^{-i}\tau_i + q^{-i-1}\tau_{i+1}$$

$$AK\tau_i = q^{-i}A\tau_i$$
$$= q^{-i}(a_i\tau_i + \tau_{i+1})$$

$$(a^{-1}K^2 + aI)\tau_i = (a^{-1}q^{-2i} + a)\tau_i$$
$$= (aq^i + a^{-1}q^{-i})q^{-i}\tau_i$$
$$= a_i q^{-i}\tau_i$$

Result follows.

(ii) Sim. $\square$

LEM 127 On V_{N-1}

$$q\Psi A - A\Psi = \frac{(q+1)ab M^{-1} - (q+ab)I}{ab-1}$$

pf For $0 \le i \le N-1$ apply each side to τ_i

$$\Psi A \tau_i = \Psi(a_i\tau_i + \tau_{i+1})$$

$$= a_i \vartheta_i \tau_{i-1} + \vartheta_{i+1}\tau_i$$

$$A\Psi\tau_i = \vartheta_i A \tau_{i-1}$$

$$= \vartheta_i(a_{i-1}\tau_{i-1} + \tau_i)$$

By L124

$$M^{-1}\tau_i = q^i\tau_i + (q-1)(a-b^{-1})\vartheta_i q^{i-1}\tau_{i-1}$$

Compare coefs for τ_i and τ_{i-1}

τ_i : $q\vartheta_{i+1} - \vartheta_i \overset{?}{=} \dfrac{(q+1)abq^i - q - ab}{ab-1}$ Use L106

τ_{i-1}: $qa_i - a_{i-1} \overset{?}{=} \dfrac{(q+1)ab(q-1)(a-b^{-1})q^{i-1}}{ab-1}$ Use L106 $\square$

LEM 128 On V_{N-1}

$$\frac{qAM^{-1} - M^{-1}A}{q-1} = (a^{-1}+b^{-1})I + (q-q^{-1})(1-a^{-1}b^{-1})\Psi$$

pf For $0 \le i \le N-1$ apply each side to τ_i

Use L124(i) and $\Psi \tau_i = \beta_i \tau_{i-1}$ $\square$

Prop 129 On V_{N-2},

$$A^2\Psi - (q+q^{-1})A\Psi A + \Psi A^2 + (q-q^{-1})^2\Psi$$

$$= \frac{1-q}{q}\,\frac{q+ab}{1-ab}\,A + \frac{(q-q^{-1})(a+b)}{1-ab}\,I$$

pf let X = either side of the equation in L127

Compute $\dfrac{qAX - XA}{q-1}$

and simplify the result using L128

Prop 130 On $V_{N,7}$,

$$\psi^2 A - (q+q^{-1})\psi A \psi + A\psi^2 = \frac{1-q}{q}\,\frac{q+ab}{1-ab}\,\psi$$

pf let X = either side of equation in L127

Compute

$$qX\psi - \psi X$$

and eval the result using

$$M\psi = q\psi M$$

□

Prop 131 On V_{N-2},

$$A^2M^{-1} - (q+q^{-1})AM^{-1}A + M^{-1}A^2 + (q-q^{-1})^2M^{-1}$$

$$= \frac{(q-1)(q-q^{-1})(q+ab)}{q\,ab} I - (a^{-1}+b^{-1})(q-1)^2 q^{-1} A$$

pf let ψ = either side of equation in L128

Compute

$$q\psi A - A\psi$$

and eval the result using L127 $\square$

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Prop 132 On V_{N+1},

$$M^{-2}A - (q+q^{-1})M^{-1}AM^{-1} + AM^{-2}$$

$$= (a^{-1}+b^{-1})(q-1)(q^{-1}-1)M^{-1}$$

pf Let $y =$ either side of eq in L128

Compute

$$qM^{-1}y - yM^{-1}$$

and simplify using

$$M\Psi = q\Psi M$$

□

Next goal: show the polynomials $\{w_i\}_{i=0}^N$ satisfy a 3-term recurrence.

Prop 133 For $0 \le i \le N-1$

$$\lambda w_i = w_{i+1} + q^{-i}(a^{-1}+b^{-1}) w_i + (1-q^{-i})(1-a^{-1}b^{-1}q^{1-i}) w_{i-1}$$

where $w_0 = 1, \quad w_{-1} = 0$

pf Ind on i

$i=0,1$ routine

$i \ge 2$ Use L 127:

$$q\Psi A - A\Psi = \frac{(q+1)\, ab M^{-1} - (q+ab) I}{ab-1}$$

Apply each side to w_i and recall

$$\Psi w_j = \theta_j w_{j-1} \qquad 0 \le j \le N$$

[Can also use L 113] $\square$

Prop 134 For $0 \le i \le N-1$,

$$\lambda w_i' = w_{i+1}' + q^i(a+b)w_i' + (1-q^i)(1-abq^{i-1})w_{i-1}'$$

where $w_0' = 1$, $w_{-1}' = 0$

pf In Prop 133 replace

$q \to q^{-1}$ $a \to a^{-1}$ $b \to b^{-1}$

$\square$

We mention another 'degenerate' case in which

$\mathcal{L} \neq 0$.

Assume $\exists\ \theta \in \mathbb{F}$ s.t.

$$a_0 \neq \theta \qquad b_0 \neq \theta$$

$$a_i = \theta \qquad b_i = \theta \qquad 1 \leq i \leq N-2$$

$$\frac{\theta - a_{N-1}}{\theta - b_0} = \frac{\theta - b_{N-1}}{\theta - a_0}$$

Obs

$$g_i = 1 \qquad 1 \leq i \leq N-1$$

$$a_{N-1} = b_0 + g_N(\theta - b_0)$$

$$b_{N-1} = a_0 + g_N(\theta - a_0)$$

For $0 \leq i \leq j \leq N-1$

$$\begin{bmatrix} j \\ i \end{bmatrix} = 1$$

Also

$$\begin{bmatrix} N \\ i \end{bmatrix} = g_N \qquad 1 \leq i \leq N-1$$

i	γ_i	η_i
0	1	1
$1 \le i \le N-1$	$(\lambda - a_0)(\lambda - \theta)^{i-1}$	$(\lambda - b_0)(\lambda - \theta)^{i-1}$
N	$(\lambda - a_0)(\lambda - \theta)^{N-2}(\lambda - a_{N-1})$	$(\lambda - b_0)(\lambda - \theta)^{N-2}(\lambda - b_{N-1})$

LEM 135 With the above assumptions,

(i) $\mathcal{L} \neq 0$

(ii) $\Delta = \dfrac{1 + \psi(\theta - b_0)}{1 + \psi(\theta - a_0)}$

pf Use th 102 $\square$