Recall our data

$$a_i = aq^i + a^{-1}q^{-i} \qquad 0 \le i \le N-1$$

$$b_i = bq^i + b^{-1}q^{-i}$$

$$a \ne b$$

$$q^i \ne 1 \qquad abq^{i-1} \ne 1 \qquad 1 \le i \le N$$

In Prop 109 we showed this data is feasible.

Next goal: show how Δ, Ψ are related via the q-exponential map $\exp_q$

Notation

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad n = 0, 1, 2, \ldots$$

$$[n]_q^! = [n]_q [n-1]_q \cdots [1]_q$$

$$\exp_q(T) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} T^n}{[n]_q^!}$$

Here it is understood T is a nilpotent linear trans on a finite dim'l vector space over $\mathbb{F}$.

One checks above $\exp_q(T)$ is invertible, with inverse

$$\exp_{q^{-1}}(-T) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{-\binom{n}{2}} T^n}{[n]_q^!}$$

LEM 110 We have

$$\left(1 - (q^2-1)T\right) \exp_q(Tq^2) = \exp_q(T)$$

pf For $n \geq 0$ compare the coef of T^n $\square$

Note

$$\exp_q(T) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)}(1-q^2)^n T^n}{(q^2;q^2)_n}$$

$$\exp_{q^{-1}}(-T) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-q^2)^n T^n}{(q^2;q^2)_n}$$

Fix. $q^{1/2} \in \mathbb{F}$ (defined up to sign)

So

$$\exp_{q^{1/2}}(T) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}}(1-q)^n T^n}{(q;q)_n}$$

$$\exp_{q^{-1/2}}(-T) = \sum_{n=0}^{\infty} \frac{(-1)^n (1-q)^n T^n}{(q;q)_n}$$

— o —

Recall the vector space

$$V = \mathrm{Span}\{\tau_i\}_{i=0}^{N} = \mathrm{Span}\{\eta_i\}_{i=0}^{N}$$

thm 111 On V,

$$\Delta = \exp_{q^{1/2}}\left(\left(a^{-1}-b\right)\psi\right) \exp_{q^{-1/2}}\left(\left(a-b^{-1}\right)\psi\right) \qquad (*)$$

pf write

$$\alpha = a^{-1}-b, \qquad \beta = b^{-1}-a$$

note $\frac{\alpha}{\beta} = \frac{b}{a}$

Recall $\psi^{N+1} = 0$ on V

For $0 \le j \le N$ compare coef of ψ^j on each side of $*$.

Require by thm 102:

$$\frac{\eta_j(a_0)}{g_1 g_2 \cdots g_j} \stackrel{?}{=} \sum_{i=0}^{j} \frac{q^{\binom{i}{2}}(1-q)^i \alpha^i}{(q;q)_i} \frac{(-1)^{j-i}(1-q)^{j-i}\beta^{j-i}}{(q;q)_{j-i}} \qquad (\star)$$

By L106

$$g_1 g_2 \cdots g_j = \frac{(q;q)_j \, (ab;q)_j}{(1-q)^j \, (1-ab)^j \, q^{\binom{j}{2}}}$$

By L108

$$\eta_j(a0) = (-1)^j b^{-j} q^{-\binom{j}{2}} (ab;q)_j \, (a^{-1}b;q)_j$$

(✱) reduces to

$$(a^{-1}b;q)_j \stackrel{?}{=} \sum_{i=0}^{j} \frac{(q^{-j};q)_i}{(q;q)_i} \left(a^{-1}b q^j\right)^i$$

Set $z = a^{-1}bq^j$

Require

$$(zq^{-j};q)_j \stackrel{?}{=} \sum_{i=0}^{j} \frac{(q^{-j};q)_i}{(q;q)_i} z^i$$

$$= {}_1\varphi_0\left(\begin{matrix} q^{-j} \\ - \end{matrix} \,\middle|\, q; z\right)$$

this is the q-binomial theorem

[see Gasper + Rahman: Basic hypergeometric series]

$\square$

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DEF 112 For $0 \le i \le N$ define

$$w_i = \exp_{q^{-1/2}}\left(\left(a - b^{-1}\right)\psi\right) \tau_i$$

$$= \exp_{q^{-1/2}}\left(\left(b - a^{-1}\right)\psi\right) \eta_i$$

$$w_i' = \exp_{q^{1/2}}\left(\left(a^{-1} - b\right)\psi\right) \tau_i$$

$$= \exp_{q^{1/2}}\left(\left(b^{-1} - a\right)\psi\right) \eta_i$$

(so w_i' is obtained from w_i by replacing
$q \to q^{-1}$, $a \to a^{-1}$ $b \to b^{-1}$)

Obs each of

$$\{w_i\}_{i=0}^{N}, \qquad \{w_i'\}_{i=0}^{N}$$

is a basis for V_0

LEM 113 For $0 \le j \le N$

$$w_j' = a^{-j}\,(ab;q)_j \sum_{i=0}^{j} \frac{(q^{-j};q)_i\,(ay;q)_i\,(ay^{-1};q)_i}{(ab;q)_i\,(q;q)_i}\, q^i$$

$$= a^{-j}\,(ab;q)_j\ {}_3\varphi_2\left(\begin{matrix} q^{-j}, & ay, & ay^{-1} \\ ab, & 0 & \end{matrix}\ \middle|\ q, q\right)$$

where $\lambda = y + y^{-1}$

To get w_j replace $q \to q^{-1}$, $a \to a^{-1}$, $b \to b^{-1}$ above

pf

$$w_j' = \exp_{q^{1/2}}\left((a^{-1}-b)\,\psi\right) \tau_j$$

$$= \sum_{i=0}^{j} \frac{q^{\binom{i}{2}}\,(1-q)^i}{(q;q)_i}\,(a^{-1}-b)^i\, \underbrace{\psi^i\, \tau_j}_{\shortparallel}$$

$$\vartheta_j \vartheta_{j-1} \cdots \vartheta_{j-i+1}\, \tau_{j-i}$$

$$\shortparallel$$

$$\frac{\vartheta_j \vartheta_{j-1} \cdots \vartheta_1}{\vartheta_1 \vartheta_2 \cdots \vartheta_{j-i}}\, \tau_{j-i}$$

$[i \to j-i]$

Now eval using L 106, L 108.

$\square$

In what follows we focus on w_i

Similar results hold for w_i'

LEM 114 We have

$$\psi w_i = \gamma_i w_{i+1} \qquad 0 \leq i \leq N$$

where $w_{N+1} = 0$

pf By Def 112 and since

$$\psi \tau_i = \gamma_i \tau_{i+1}$$

$\square$

DEF 115 Define

$$K, B, M \in \mathrm{End}(V)$$

by

$$K \tau_i = q^{-i} \tau_i \qquad 0 \le i \le N$$

$$B \eta_i = q^{-i} \eta_i \qquad ''$$

$$M w_i = q^{-i} w_i \qquad ''$$

LEM 116

(i) $K \Psi = q \Psi K$

(ii) $B \Psi = q \Psi B$

(iii) $M \Psi = q \Psi M$

pf (i) For $0 \le i \le N$

$$K \Psi \tau_i = \mathfrak{J}_i K \tau_{i-1} = \mathfrak{J}_i q^{1-i} \tau_{i-1}$$

and

$$q \Psi K \tau_i = q^{1-i} \Psi \tau_i = \mathfrak{J}_i q^{1-i} \tau_{i-1}$$

(ii), (iii) Sim

$\square$

LEM 117

(i) $\quad B\Delta = \Delta K$

(ii) $\quad K \exp_{q^{1/2}}\left((b^{-1}-a)\psi\right) = \exp_{q^{1/2}}\left((b^{-1}-a)\psi\right) M$

(iii) $\quad B \exp_{q^{1/2}}\left((a^{-1}-b)\psi\right) = \exp_{q^{1/2}}\left((a^{-1}-b)\psi\right) M$

pf (i) Apply each side to τ_i for $0 \le i \le N$

(ii), (iii) Apply each side to w_i for $0 \le i \le N$ $\square$

Prop 118

(i) $$M^{-1} = K^{-1}\Big(1 + (q-1)(a-b^{-1})\,\Psi\Big)$$

(ii) $$M^{-1} = \Big(1 + (q^{-1}-1)(b^{-1}-a)\,\Psi\Big)K^{-1}$$

(iii) $$M^{-1} = B^{-1}\Big(1 + (q-1)(b-a^{-1})\,\Psi\Big)$$

(iv) $$M^{-1} = \Big(1 + (q^{-1}-1)(a^{-1}-b)\,\Psi\Big)B^{-1}$$

pf (i) Recall

$$K\Psi K^{-1} = q\Psi$$

Define

$$T = (b^{-1}-a)\Psi$$

so

$$KTK^{-1} = qT$$

By L 110

$$\big(1-(q-1)T\big)\exp_{q^{1/2}}(qT) = \exp_{q^{1/2}}(T)$$

obs

$$\exp_{q^{1/2}}(qT) = \exp_{q^{1/2}}(KTK^{-1})$$

$$= K \exp_{q^{1/2}}(T) K^{-1}$$

So

$$(1-(q-1)T) K \exp_{q^{1/2}}(T) K^{-1} = \exp_{q^{1/2}}(T)$$

$$(1-(q-1)T)\underbrace{\left(\exp_{q^{1/2}}(T)\right)^{-1} K \exp_{q^{1/2}}(T) K^{-1}}_{\overset{\text{L117(ii)}}{=} M} = I$$

(ii) Eval (i) using $K\Psi = q\Psi K$

(iii), (iv) Similar

$\square$

LEM 119

$$M = \frac{bK - aB}{b-a}$$

pf By Prop 118

$$K = \left(1 + (q-1)(a - b^{-1})\psi\right) M$$

$$B = \left(1 + (q-1)(b - a^{-1})\psi\right) M$$

So $bK - aB = (b-a)M$ $\square$

We now eliminate M in Prop 118
to see how K, B, ψ are related.

LEM 120

$$KB^{-1} = \frac{1+(q-1)(a-b^{-1})\psi}{1+(q-1)(b-a^{-1})\psi}$$

$$BK^{-1} = \frac{1+(q-1)(b-a^{-1})\psi}{1+(q-1)(a-b^{-1})\psi}$$

$$K^{-1}B = \frac{1+(q^{-1}-1)(a^{-1}-b)\psi}{1+(q^{-1}-1)(b^{-1}-a)\psi}$$

$$B^{-1}K = \frac{1+(q^{-1}-1)(b^{-1}-a)\psi}{1+(q^{-1}-1)(a^{-1}-b)\psi}$$

pf use Prop 118

$\square$

cw

LEM 121

$$\psi = \frac{1}{q-1} \frac{1}{1-ab} \frac{I - K B^{-1}}{b^{-1} I - a^{-1} K B^{-1}}$$

$$\psi = \frac{1}{q-1} \frac{1}{1-ab} \frac{I - B K^{-1}}{a^{-1} I - b^{-1} B K^{-1}}$$

$$\psi = \frac{1}{q^{-1}-1} \frac{1}{1-a^{-1}b^{-1}} \frac{I - K^{-1} B}{b I - a K^{-1} B}$$

$$\psi = \frac{1}{q^{-1}-1} \frac{1}{1-a^{-1}b^{-1}} \frac{I - B^{-1} K}{a I - b B^{-1} K}$$

pf In L 120 solve for ψ $\square$

thm 122 We have

$$0 = aB^2 - \frac{bq-a}{q-1} BK - \frac{aq-b}{q-1} KB + bK^2,$$

$$0 = aK^{-2} - \frac{bq-a}{q-1} B^{-1}K^{-1} - \frac{aq-b}{q-1} K^{-1}B^{-1} + bB^{-2}.$$

pf In L121 elim 4

thm 123 We have

$$\frac{qM^{-1}K - KM^{-1}}{q-1} = I$$

$$\frac{qM^{-1}B - BM^{-1}}{q-1} = I$$

pf To verify the above equations, eliminate M using L119 and compare the results with Th 122. $\square$