Lecture 25 Wednesday Oct 30

Recall data

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1}$$

and

$$V = \mathrm{Span}\{\tau_i\}_{i=0}^{N} = \mathrm{Span}\{\eta_i\}_{i=0}^{N}$$

$$\mathcal{L} = \{ \varphi \in \mathrm{End}(V) \mid \varphi \text{ is } DL \}$$

thm 102 TFAE

(i) $\mathcal{L} \neq 0$

(ii) For $0 \le j \le N$,

$$\eta_j = \sum_{i=0}^{j} \eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_g \tau_i \qquad (1)$$

(iii) For $0 \le j \le N$,

$$\tau_j = \sum_{i=0}^{j} \tau_{j-i}(b_0) \begin{bmatrix} j \\ i \end{bmatrix}_g \eta_i \qquad (2)$$

Suppose (i)–(iii) hold, and let $\psi \in \mathcal{L}$ be normalized.

Then

$$\Delta = \sum_{i=0}^{N} \frac{\eta_i(a_0)}{g_1 g_2 \cdots g_i} \psi^i \qquad (3)$$

$$\Delta^{-1} = \sum_{i=0}^{N} \frac{\tau_i(b_0)}{g_1 g_2 \cdots g_i} \psi^i \qquad (4)$$

pf $(i) \to (ii)$ Let j be given

$\exists$ scalars $\{\alpha_i\}_{i=0}^{j}$ in $\mathbb{F}$ s.t.

$$\tau_j = \sum_{i=0}^{j} \alpha_i \tau_i \qquad (*)$$

For $0 \le i \le j$ find α_i.

Let $\psi \in \mathcal{L}$ be normalized. In $*$ apply ψ^i to each side, and evaluate the result at $\lambda = a_0$

Use Lem 100. Also recall that for $0 \le n \le N$

$\tau_n(a_0) = 1$ if $n = 0$ and $\tau_n(a_0) = 0$ if $n \ne 0$.

By these comments

$$g_j g_{j-1} \cdots g_{j-i+1} \tau_{j-i}(a_0) = \alpha_i \, g_1 g_2 \cdots g_i$$

So

$$\alpha_i = \tau_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_g$$

So (i) holds.

$(ii) \to (i)$ Define $\psi \in \mathrm{End}(V)$ s.t.

$$\psi(\tau_i) = \vartheta_i \tau_{i-1} \qquad \text{for } 0 \le i \le N$$

obs $\psi \ne 0$ since $N \ge 1$, $\vartheta_1 = 1$, $\tau_0 = 1$

Show $\psi \in \mathcal{L}$.

<u>Claim</u> ψ satisfies (3)

<u>pf cl</u> For $0 \le j \le N$ show each side of (3) sends $\tau_j \to \eta_j$.

LHS: $\Delta \tau_j = \eta_j$ by def of Δ ✓

RHS:

$$\sum_{i=0}^{j} \frac{\eta_i(a_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_i} \psi^i \tau_j = \sum_{i=0}^{j} \frac{\eta_i(a_0)}{\vartheta_1 \vartheta_2 \cdots \vartheta_i} \vartheta_j \vartheta_{j-1} \cdots \vartheta_{j-i+1} \tau_{j-i}$$

$$= \sum_{i=0}^{j} \eta_i(a_0) \left[\begin{matrix} j \\ i \end{matrix} \right]_\vartheta \tau_{j-i}$$

$$[\, i \to j-i \,]$$

$$= \sum_{i=0}^{j} \eta_{j-i}(a_0) \left[\begin{matrix} j \\ i \end{matrix} \right]_\vartheta \tau_i$$

$$= \eta_j \quad ✓$$

By the claim,

$$\Psi \Delta = \Delta \Psi$$

Now $\Psi \in \mathcal{L}$ so $\mathcal{L} \neq 0$

(i) $\Leftrightarrow$ (iii) Interchange $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$

in the proof of (i) $\Leftrightarrow$ (ii)

Now assume (i)–(iii) hold.

We saw in the proof of (ii) $\rightarrow$ (i) that (3) holds.

Interchaning the roles of $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ we

obtain (4). $\square$

We now find all

$\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$

that satisfy the conditions in Th 102

First we consider some trivial solutions.

LEM 103

(i) Assume $N \le 2$, then $\mathcal{L} \ne 0$.

(ii) Assume $N = 3$. then $\mathcal{L} \ne 0$ iff

$$(a_0 - b_1)(b_1 - a_2) = (b_0 - a_1)(a_1 - b_2)$$

pf (i) Check thm 102 (ii)

$i=0$: $\quad \eta_0 = 1 = \tau_0$

$i=1$:

$$\underbrace{\eta_1}_{\lambda - b_0} \stackrel{?}{=} \underbrace{\eta_1(a_0)}_{a_0 - b_0} \underbrace{\tau_0}_{1} + \underbrace{\tau_1}_{\lambda - a_0}$$

✓

$i=2$:

$$\eta_2 \stackrel{?}{=} \eta_2(a_0)\tau_0 + \eta_1(a_0)\frac{\gamma_2}{\gamma_1}\tau_1 + \tau_2$$

$$(\lambda - b_0)(\lambda - b_1) \stackrel{?}{=} (a_0 - b_0)(a_0 - b_1)$$

$$+ (a_0 - b_0)\frac{a_0 + a_1 - b_0 - b_1}{a_0 - b_0}(\lambda - a_0)$$

$$+ (\lambda - a_0)(\lambda - a_1)$$

compare coefs of $1, \lambda, \lambda^2$

it works ✓

(ii) Require

$\eta_3 =$

term	coef
τ_0	$\eta_3(a_0)$
τ_1	$\eta_2(a_0)\ \frac{\gamma_3}{\gamma_1}$
τ_2	$\eta_1(a_0)\ \frac{\gamma_3}{\gamma_1}$
τ_3	1

$$(\lambda - b_0)(\lambda - b_1)(\lambda - b_2)$$

$$\stackrel{?}{=} (a_0 - b_0)(a_0 - b_1)(a_0 - b_2)$$

$$+ (a_0 - b_0)(a_0 - b_1)(\lambda - a_0) \frac{a_0 + a_1 + a_2 - b_0 - b_1 - b_2}{a_0 - b_0}$$

$$+ (a_0 - b_0)(\lambda - a_0)(\lambda - a_1) \frac{a_0 + a_1 + a_2 - b_0 - b_1 - b_2}{a_0 - b_0}$$

$$+ (\lambda - a_0)(\lambda - a_1)(\lambda - a_2)$$

Compare coefs $1, \lambda, \lambda^2, \lambda^3$ (ex)

$\square$

We mention some 'degenerate' cases in which $\mathcal{L} \neq 0$.

LEM 104 Assume

$$a_{i-1} = b_i \qquad 1 \leq i \leq N-1$$

Then

(i) $\mathcal{L} \neq 0$

(ii) $$\vartheta_i = \frac{a_{i-1} - b_0}{a_0 - b_0} \qquad 1 \leq i \leq N$$

(iii) $$\gamma_i = (\lambda - b_0)\, \tau_{i-1} \qquad 1 \leq i \leq N$$

(iv) $$\eta_i(a_0) = 0 \qquad 2 \leq i \leq N$$

(v) $$\Delta = I + (a_0 - b_0)\, \Psi$$

pf (ii) – (iv) clear

(i) Apply Th 102 (ii)

(ii) By Line (3) in Th 102

$\square$

LEM 105 Assume

$$a_i = b_{i-1} \qquad 1 \le i \le N-1$$

then

(i) $\mathcal{L} \neq 0$

(ii) $$\gamma_i = \frac{a_0 - b_{i-1}}{a_0 - b_0} \qquad 1 \le i \le N$$

(iii) $$\tau_i = (\lambda - a_0)\,\eta_{i-1} \qquad 1 \le i \le N$$

(iv) $$\tau_i(b_0) = 0 \qquad 2 \le i \le N$$

(v) $$\Delta^{-1} = I + (b_0 - a_0)\,\Psi$$

pf Interchange $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ in L104 $\square$

We now describe the "most general" case in which $\mathcal{L} \neq 0$

Until further notice:

Fix nonzero $a, b, q \in \mathbb{F}$ and define

$$a_i = a q^i + a^{-1} q^{-i} \qquad 0 \le i \le N-1$$

$$b_i = b q^i + b^{-1} q^{-i} \qquad \cdots$$

Assume

$$a \neq b$$

and

$$q^i \neq 1 \qquad a b q^{i-1} \neq 1 \qquad 1 \le i \le N$$

So that

$$a_0 + \cdots + a_{i-1} \neq b_0 + \cdots + b_{i-1} \qquad 1 \le i \le N$$

Next goal: Show the above $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ is feasible.

Obs that for $0 \le i \le N$

$$a_0 + a_1 + \cdots + a_{i-1} = \frac{q^i - 1}{q - 1} \left(a + a^{-1} q^{1-i} \right)$$

$$b_0 + b_1 + \cdots + b_{i-1} = \frac{q^i - 1}{q - 1} \left(b + b^{-1} q^{1-i} \right)$$

LEM 106 For $0 \le i \le N$

$$\vartheta_i = \frac{1-q^i}{1-q} \frac{1-abq^{i+1}}{1-ab} q^{1-i}$$

$$= \frac{1-q^{-i}}{1-q^{-1}} \frac{1-a^{-1}b^{-1}q^{-1-i}}{1-a^{-1}b^{-1}} q^{i-1}$$

pf By Def 95 $\square$

Recall notation

$$(\alpha;q)_n = (1-\alpha)(1-\alpha q) \cdots (1-\alpha q^{n-1}) \qquad n=0,1,2,\ldots$$

Interp $(\alpha;q)_0 = 1.$

LEM 107 For $0 \le i \le j \in \mathbb{N}$,

$$\begin{bmatrix} j \\ i \end{bmatrix}_g = \frac{(q^{-j};q)_i \, (a^{-1}b^{-1}q^{1-j};q)_i \, q^{ij}(ab)^i}{(q;q)_i \, (ab;q)_i}$$

pf Use L106 (Details below)

$g_1 g_2 \cdots g_i =$

$$\frac{1-q}{1-q} \quad \frac{1-q^2}{1-q} \quad \cdots \quad \frac{1-q^i}{1-q}$$

$\times$

$$\frac{1-ab}{1-ab} \quad \frac{1-abq}{1-ab} \quad \cdots \quad \frac{1-abq^{i-1}}{1-ab}$$

$\times$

$$q^0 \quad q^{-1} \quad q^{-2} \quad \cdots \quad q^{1-i}$$

$$= \frac{(q;q)_i \, (ab;q)_i}{(1-q)^i (1-ab)^i} \, q^{-\binom{i}{2}}$$

$$\partial_J \partial_{J-1} \cdots \partial_{J-i+1} =$$

$$\frac{1-q^{-J}}{1-q^{-1}} \quad \frac{1-q^{1-J}}{1-q^{-1}} \quad \cdots \quad \frac{1-q^{i-1-J}}{1-q^{-1}}$$

$$\times$$

$$\frac{1-a^{-1}b^{-1}q^{1-J}}{1-a^{-1}b^{-1}} \quad \frac{1-a^{-1}b^{-1}q^{2-J}}{1-a^{-1}b^{-1}} \quad \cdots \quad \frac{1-a^{-1}b^{-1}q^{i-J}}{1-a^{-1}b^{-1}}$$

$$\times \quad q^{J-1} \quad q^{J-2} \quad \cdots \quad q^{J-i}$$

$$= \frac{(q^{-J};q)_i \,(a^{-1}b^{-1}q^{1-J};q)_i}{(1-q^{-1})^i \,(1-a^{-1}b^{-1})^i} \quad q^{i(J-i)} \quad q^{\binom{i}{2}}$$

Also

$$1-q^{-1} = -q^{-1}(1-q)$$

$$(1-q^{-1})^i = (-1)^i q^{-i}(1-q)^i$$

$$1-a^{-1}b^{-1} = -(1-ab)\,a^{-1}b^{-1}$$

$$(1-a^{-1}b^{-1})^i = (-1)^i (1-ab)^i a^{-i}b^{-i}$$

$\square$

LEM 108 For $0 \le i \le N$

(i) $$\tau_i(\lambda) = (-1)^i a^{-i} q^{-\binom{i}{2}} (ay;q)_i (ay^{-1};q)_i$$

(ii) $$\eta_i(\lambda) = (-1)^i b^{-i} q^{-\binom{i}{2}} (by;q)_i (by^{-1};q)_i$$

where $\lambda = y + y^{-1}$

pf (i) For $1 \le j \le N$,

$$\lambda - a_{j-1} = y + y^{-1} - aq^{j-1} - a^{-1}q^{1-j}$$

$$= -a^{-1}q^{1-j}(1 - ay q^{j-1})(1 - ay^{-1}q^{j-1})$$

Result follows.

(ii) Sim $\square$

Prop 109 $\mathcal{L} \neq 0$

pf We show th 102 (ii) holds.

For $0 \le j \le N$ show

$$\eta_j = \sum_{i=0}^{j} \eta_{j-i}(a_0) \begin{bmatrix} j \\ i \end{bmatrix}_q \tau_i \qquad (*)$$

By L108 (ii)

$$\eta_j = (-1)^j b^{-j} q^{-\binom{j}{2}} (by;q)_j (by^{-1};q)_j$$

For $0 \le i \le j$

$$\eta_{j-i}(a_0) = (-1)^{j-i} b^{i-j} q^{-\binom{j-i}{2}} (a\cdot b;q)_{j-i} (a^{-1}b;q)_{j-i}$$

By L 107

$$\begin{bmatrix} j \\ i \end{bmatrix}_q = \frac{(q^{-j};q)_i (a^{-1}b^{-1}q^{1-j};q)_i q^{ij} a^i b^i}{(q;q)_i (ab;q)_i}$$

$$\tau_i = (-1)^i a^{-i} q^{-\binom{i}{2}} (ay;q)_i (ay^{-1};q)_i$$

To verify * WLOG view a, b as indeterminates instead of scalars.

For $0 \le i \le d$

$$(ab;q)_{d-i} = \frac{(ab;q)_d \, (-1)^i \, q^{\binom{i+1}{2}}}{(a^{-1}b^{-1}q^{1-d};q)_i \, a^i \, b^i \, q^{id}}$$

$$(a^{-1}b;q)_{d-i} = \frac{(a^{-1}b;q)_d \, (-1)^i \, q^{\binom{i+1}{2}} \, a^i}{(ab^{-1}q^{1-d};q)_i \, b^i \, q^{id}}$$

the desired equation (*) becomes

$$\frac{(by;q)_t\,(by^{-1};q)_t}{(ab;q)_t\,(a^{-1}b;q)_t}$$

$$= \sum_{i=0}^{t} \frac{(q^{-t};q)_i\,(ay;q)_i\,(ay^{-1};q)_i\; q^i}{(ab;q)_i\,(ab^{-1}q^{1-t};q)_i\,(q;q)_i}$$

$$= {}_3\varphi_2\left(\begin{matrix} q^{-t}, & ay, & ay^{-1} \\ & ab & ab^{-1}q^{1-t} \end{matrix} \Big/ \; q; q\right)$$

this is an instance of the q-Saalschutz identity [Gasper + Rahman's book Basic hypergeom series]

We have shown * so $\mathcal{L} \neq 0$ □