LECTURE 24 MONDAY OCT 28

Next topic: Double Lowering maps

Motivation Given TD system Φ on V with Φ-split decomp $\{U_i\}_{i=0}^d$ it turns out $\exists\ \psi \in \mathrm{End}(V)$ s.t. $\psi U_i \subseteq U_{i-1}$ and $\psi U_i^{\Downarrow} \subseteq U_{i-1}^{\Downarrow}$ for $0 \le i \le d$

To understand such ψ it is helpful to consider a more general setting.

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Until further notice

$\mathbb{F}$ = alg closed field

N = positive integer or ∞

Given two sequences of scalars from $\mathbb{F}$, denoted

$$\{a_i\}_{i=0}^{N-1}, \qquad \{b_i\}_{i=0}^{N-1} \qquad (*)$$

To avoid degenerate situations we assume

$$a_0 + a_1 + \cdots + a_{i-1} \neq b_0 + b_1 + \cdots + b_{i-1} \qquad 1 \le i \le N$$

Let λ denote an indeterminate

Let $\mathbb{F}[\lambda] =$ $\mathbb{F}$-algebra of all polys in λ that have all coeffs in $\mathbb{F}$

For $0 \le i \le N$ define $\tau_i, \eta_i \in \mathbb{F}[\lambda]$ by

$$\tau_i = (\lambda - a_0)(\lambda - a_1) \cdots (\lambda - a_{i-1})$$

$$\eta_i = (\lambda - b_0)(\lambda - b_1) \cdots (\lambda - b_{i-1})$$

So τ_i, η_i are monic of deg i.

Define

$$V = \mathrm{Span}\{\lambda^i\}_{i=0}^{N} \subseteq \mathbb{F}[\lambda]$$

Obs each of

$$\{\tau_i\}_{i=0}^{N}, \qquad \{\eta_i\}_{i=0}^{N}$$

is a basis for V.

For notational convenience set

$$\tau_{-1} = 0, \qquad \eta_{-1} = 0$$

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Def 91 For $\psi \in End(V)$

ψ is <u>Double Lowering</u> (DL) with respect to

(*) whenever both

$\psi \tau_i \in \mathbb{F}\tau_{i-1}$ $\qquad$ $\psi \eta_i \in \mathbb{F}\eta_{i-1}$

for $0 \leq i \leq N$.

DEF 92 Let

$$\mathcal{L} = \{ \psi \in End(V) \mid \psi \text{ is DL rel } * \}$$

$\mathcal{L}$ is subspace of $\mathbb{F}$-vector space $End(V)$

Call $\mathcal{L}$ the <u>Double Lowering space</u> for (*).

Call (*) <u>feasible</u> whenever $\mathcal{L} \neq 0$

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Our goal: Find nec/suf conditions on *

for $\mathcal{L} \neq 0$. In this case describe $\mathcal{L}$.

Remarks

For $1 \le i \le N$

- in τ_i the coef of λ^{i-1} is

$$-a_0 - a_1 - \cdots - a_{i-1}$$

- in η_i the coef of λ^{i-1} is

$$-b_0 - b_1 - \cdots - b_{i-1}$$

For $0 \le n \le N$ define

$$V_n = \operatorname{Span}\{\lambda^i\}_{i=0}^{n}$$

So $$V_N = V$$

For $0 \le n \le N$, each of

$$\{\tau_i\}_{i=0}^{n}, \qquad \{\eta_i\}_{i=0}^{n}$$

is a basis for V_n

For notational convenience let

$$V_{-1} = 0$$

LEM 93 For $1 \le i \le N$,

(i) $\eta_i - \tau_i \in V_{i-1}$

(ii) the coef of λ^{i-1} in $\eta_i - \tau_i$ is

$$a_0 + a_1 + \cdots + a_{i-1} - b_0 - b_1 - \cdots - b_{i-1}$$

pf clear. $\square$

Define $\Delta \in \mathrm{End}(V)$ s/t

$$\Delta \tau_i = \eta_i \qquad 0 \le i \le N$$

obs Δ^{-1} exists

obs

$$\Delta V_n = V_n \qquad 0 \le n \le N$$

Define a lin trans

$$A : \begin{array}{l} V_{N-1} \to V_N \\ v \to \lambda v \end{array}$$

obs

$$A V_{n-1} \subseteq V_n \qquad 1 \le n \le N$$

obs

$$\begin{aligned} A \tau_i &= a_i \tau_i + \tau_{i+1} \qquad 0 \le i \le N-1 \\ A \eta_i &= b_i \eta_i + \eta_{i+1} \end{aligned}$$

Note for $0 \le i, j \le N$

$$\tau_i(a_j) = 0, \qquad \eta_i(b_j) = 0 \quad \text{if } i > j$$

LEM 94 Assume $(*)$ is feasible. Given $s, t \in \mathbb{F}$ with $s \neq 0$. Then the pair

$$\{s a_i + t\}_{i=0}^{N}, \qquad \{s b_i + t\}_{i=0}^{N} \tag{1}$$

is feasible.

pf For $\alpha \in \mathbb{F}$ define

$$\alpha' = s\alpha + t$$

For $0 \le i \le N$ define

$$\tau_i' = (\lambda - a_0')(\lambda - a_1') \cdots (\lambda - a_{i-1}')$$
$$\eta_i' = (\lambda - b_0')(\lambda - b_1') \cdots (\lambda - b_{i-1}')$$

Let $\mathcal{L}'$ denote DL space for (1). Show $\mathcal{L}' \neq 0$

Define $$\bar{\lambda} = \frac{\lambda - t}{s}$$

So $\{\bar{\lambda}^i\}_{i=0}^N$ is basis for V

So $\exists$ iso of $\mathbb{F}$-vector spaces

$$\mathcal{E} \quad \begin{aligned} V &\to V \\ \lambda^i &\to \bar{\lambda}^i \end{aligned}$$

Note that

$$\lambda - \alpha' = s(\bar{\lambda} - \alpha) \qquad \alpha \in \mathbb{F}$$

This gives

$$\mathcal{E}(\tau_i) = s^{-1} \tau_i' \qquad 0 \le i \le N$$

$$\mathcal{E}(\eta_i) = s^{-1} \eta_i'$$

Now the map

$$\begin{aligned} \mathcal{L} &\to \mathcal{L}' \\ \varphi &\to \mathcal{E} \varphi \mathcal{E}^{-1} \end{aligned}$$

is an iso of $\mathbb{F}$-vector spaces. We assume $\mathcal{L} \neq 0$

so $\mathcal{L}' \neq 0$. $\square$

Note: By L 94, for a feas pair $\{a_i\}_{i=0}^{N-1}$, $\{b_i\}_{i=0}^{N-1}$ the scalars a_0, b_0 are "free". Given any distinct a_0', b_0' in $\mathbb{F}$ define

$$s = \frac{a_0' - b_0'}{a_0 - b_0} \qquad t = \frac{a_0 b_0' - a_0' b_0}{a_0 - b_0}$$

then

$$s a_0 + t = a_0' \qquad s b_0 + t = b_0'$$

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DEF 95 For $0 \le i \le N$ define

$$\vartheta_i = \frac{a_0 + a_1 + \cdots + a_{i-1} - b_0 - b_1 - \cdots - b_{i-1}}{a_0 - b_0}$$

So $\vartheta_0 = 0$ and $\vartheta_1 = 1$

obs

$$\vartheta_i \neq 0 \qquad 1 \le i \le N$$

Prop 96 For $\psi \in \mathcal{L}$ and $1 \le i \le N$,

(i) In $\psi(\lambda^i)$ the coef of λ^{i-1} is $\psi(\lambda)\, g_i$

(ii) $\psi(\tau_i) = \psi(\lambda)\, g_i\, \tau_{i-1}$

(iii) $\psi(\eta_i) = \psi(\lambda)\, g_i\, \eta_{i-1}$

pf Use induction on i

Case $i=1$ trivial

Case $i \ge 2$:

obs $\psi(\lambda^i) \in V_{i-1}$

Let $\alpha_i =$ coef of λ^{i-1} in $\psi(\lambda^i)$

By assumption
$$\psi(\tau_i) \in \mathbb{F}\, \tau_{i-1}$$

Each of τ_i, τ_{i-1} monic so

$$\psi(\tau_i) = \alpha_i\, \tau_{i-1}$$

Similarly
$$\psi(\eta_i) = \alpha_i\, \eta_{i-1}$$

So

$$\psi(\eta_i - \tau_i) = \alpha_i (\eta_{i-1} - \tau_{i-1}) \qquad (1)$$

Show $\qquad \alpha_i = \psi(\lambda)\, g_i$

By L93

$$\eta_i - \tau_i - (a_0 - b_0)\, g_i\, \lambda^{i-1} \in V_{i-2}$$

In this inclusion apply ψ to each side, to get

$$\psi(\eta_i - \tau_i) - (a_0 - b_0)\, g_i\, \psi(\lambda^{i-1}) \in V_{i-3} \qquad (2)$$

Note that

$$\eta_{i-1} - \tau_{i-1} - (a_0 - b_0)\, g_{i-1}\, \lambda^{i-2} \in V_{i-3} \qquad (3)$$

By induction

$$\psi(\lambda^{i-1}) - \psi(\lambda)\, g_{i-1}\, \lambda^{i-2} \in V_{i-3} \qquad (4)$$

Combine (1)—(4) as follows

~~eq~~	(1)	(2)	(3)	(4)
coef	1	-1	α_i	$-(a_0-b_0)\,\mathcal{J}_i$

this shows that

$$(a_0-b_0)\,\mathcal{J}_{i-1}\,\lambda^{i-2} \qquad (5)$$

times

$$\alpha_i - \psi(\lambda)\,\mathcal{J}_i \qquad (6)$$

is contained in V_{i-3}.

By assumption $a_0-b_0 \neq 0$ and $\mathcal{J}_{i-1} \neq 0$ so (5) is not in V_{i-3}. therefore (6) is zero so

$$\alpha_i = \psi(\lambda)\,\mathcal{J}_i$$

$\square$

COR 97 $\mathcal{L} = 0$ or $\dim \mathcal{L} = 1$

COR 98 For $\psi \in \mathcal{L}$

$$\psi = 0 \quad \text{iff} \quad \psi(\lambda) = 0$$

Def 99 Assume $(*)$ is feasible, so $\dim \mathcal{L} = 1$

By Cor 97, 98 $\exists$ unique $\psi \in \mathcal{L}$ with

$$\psi(\lambda) = 1.$$

Call this ψ <u>normalized.</u>

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LEM 100 For $\psi \in \operatorname{End}(V)$ TFAE

(i) For $0 \le i \le N$ both

$$\psi(\tau_i) = \mathfrak{z}_i \tau_{i+1}, \qquad \psi(\eta_i) = \mathfrak{z}_i \eta_{i+1}$$

(ii) $\mathcal{L} \neq 0$ and $\psi \in \mathcal{L}$ is normalized.

pf By Def 99 and Prop 96 $\square$

Prop 101 For $\psi \in End(V)$ TFAE

(i) $\Delta \psi = \psi \Delta$ and

$\psi(\tau_i) \in \mathbb{F}\, \tau_{i-1}$ for $0 \le i \le N$ (1)

(ii) $\Delta \psi = \psi \Delta$ and

$\psi(\eta_i) \in \mathbb{F}\, \eta_{i-1}$ for $0 \le i \le N$ (2)

(iii) $\psi \in \mathcal{L}$

pf (i) → (ii) Show (2)

$$\psi \eta_i = \psi \Delta \tau_i$$
$$= \Delta \psi \tau_i$$
$$\in \Delta \mathbb{F}\, \tau_{i-1}$$
$$= \mathbb{F}\, \eta_{i-1}$$

(ii) → (i) Sim

(i), (ii) → (iii) Def of $\mathcal{L}$

(iii) → (i) WLOG ψ is normalized

So $\psi(\tau_i) = g_i \tau_{i+1}$ $\psi(z_i) = g_i z_{i+1}$ $0 \le i \le N$

So for $0 \le i \le N$

$$\Delta \psi \tau_i = g_i \Delta \tau_{i+1} = g_i z_{i+1} = \psi z_i = \psi \Delta \tau_i$$

So

$$\Delta \psi = \psi \Delta$$

By def ψ satisfies (1) $\square$

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Notation: For $0 \le i \le j \le N$ define

$$\begin{bmatrix} j \\ i \end{bmatrix}_g = \frac{g_j g_{j-1} \cdots g_{j-i+1}}{g_1 g_2 \cdots g_i}$$

$$= \frac{g_j g_{j-1} \cdots g_{i+1}}{g_1 g_2 \cdots g_{j-i}}$$