LECTURE 23 Friday Oct 25

Next goal

- show each TD pair A, A^* satisfies the tridiagonal relations.

Until further notice:

$\mathbb{F}$ = any field

Fix integer $d \geq 0$

$\{\theta_i\}_{i=0}^d$ is any sequence of scalars from $\mathbb{F}$

DEF 82 Given $\beta, \gamma, \delta \in \mathbb{F}$

(i) $\{\theta_i\}_{i=0}^d$ is called <u>recurrent</u> whenever

$\theta_{i-1} \neq \theta_i$ for $1 \leq i \leq d$, and

$$\frac{\theta_{i-2} - \theta_{i+1}}{\theta_{i-1} - \theta_i} \qquad (*)$$

is indep of i for $2 \leq i \leq d-1$

(ii) $\{\theta_i\}_{i=0}^{d}$ is called <u>β-recurrent</u> whenever

$$\theta_{i-2} - (\beta+1)\theta_{i-1} + (\beta+1)\theta_i - \theta_{i+1} = 0$$

for $2 \le i \le d-1$

(iii) $\{\theta_i\}_{i=0}^{d}$ is called <u>(β,γ)-recurrent</u> whenever

$$\theta_{i-1} - \beta\theta_i + \theta_{i+1} = \gamma$$

for $1 \le i \le d-1$

(iv) $\{\theta_i\}_{i=0}^{d}$ is called <u>(β,γ,δ)-recurrent</u> whenever

$$\theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1}+\theta_i) = \delta$$

for $1 \le i \le d$

— o —

The above notions are related as follows.

LEM 83 TFAE

(i) $\{\theta_i\}_{i=0}^{d}$ is rec

(ii) $\exists\ \beta \in \mathbb{F}$ s.t $\{\theta_i\}_{i=0}^{d}$ is βrec, and

$\theta_{i-1} \neq \theta_i$ for $2 \le i \le d-1$

Suppose (i)–(ii) hold, and $d \geq 3$. Then

$\beta + 1$ = common value of (*)

pf clear $\square$

LEM 84 Given $\beta \in \mathbb{F}$ TFAE

(i) $\{\theta_i\}_{i=0}^{d}$ is β-rec

(ii) $\exists\ \gamma \in \mathbb{F}$ s.t $\{\theta_i\}_{i=0}^{d}$ is (β, γ)-rec

pf routine $\square$

LEM 85 Given $\beta, \gamma \in \mathbb{F}$.

(i) Assume $\{\theta_i\}_{i=0}^{d}$ is (β,γ)-rec. Then
$\exists\, \delta \in \mathbb{F}$ s.t. $\{\theta_i\}_{i=0}^{d}$ is (β,γ,δ)-rec.

(ii) For $\delta \in \mathbb{F}$, assume $\{\theta_i\}_{i=0}^{d}$ is (β,γ,δ)-rec,
and $\theta_{i-1} \neq \theta_{i+1}$ for $1 \le i \le d-1$. Then
$\{\theta_i\}_{i=0}^{d}$ is (β,γ)-rec.

$\square$

pf Similar to the pf of L 21, 22.

— o —

Recall $\overline{\mathbb{F}}$ = alg closure of $\mathbb{F}$

thm 86 Given $\beta \in \mathbb{F}$ and assume $\{\theta_i\}_{i=0}^d$ is β-rec.

Case I $\beta \neq 2$, $\beta \neq -2$

$$\theta_i = a + bq^i + cq^{-i} \qquad 0 \le i \le d$$

$q, a, b, c \in \overline{\mathbb{F}}$ $\qquad q + q^{-1} = \beta$

$q \neq 0$, $q \neq 1$, $q \neq -1$

Case II $\beta = 2$, $\mathrm{char}(\mathbb{F}) \neq 2$

$$\theta_i = a + bi + ci^2 \qquad 0 \le i \le d$$

$a, b, c \in \mathbb{F}$

Case III $\beta = -2$, $\mathrm{char}(\mathbb{F}) \neq 2$

$$\theta_i = a + b(-1)^i + ci(-1)^i \qquad 0 \le i \le d$$

$a, b, c \in \mathbb{F}$

Case IV $\beta = 0$, $\mathrm{char}(\mathbb{F}) = 2$

$$\theta_i = a + bi + c\left(\tfrac{i}{2}\right) \qquad 0 \le i \le d$$

where $\left(\frac{i}{2}\right) = \begin{cases} 0 & \text{if } i = 0 \text{ or } i = 1 \pmod 4 \\ 1 & \text{if } i = 2 \text{ or } i = 3 \pmod 4 \end{cases}$

$a, b, c \in \mathbb{F}$

Pf Routine $\square$

Now we bring in A, A^*.

Let V denote a vector space $/\mathbb{F}$ with fin pos dim.

Given $A, A^* \in \mathrm{End}(V)$

Assume A is diagonalizable, with eigvals $\{\theta_i\}_{i=0}^d$

For $0 \le i \le d$ let E_i = prim idemp of A for θ_i

Let M denote the subalgebra of $\mathrm{End}(V)$ gen by A.

LEM 87 Assume

$$E_i A^* E_j = 0 \quad \text{if} \quad |i-j|>1 \qquad (0 \le i,j \le d)$$

then

$$\mathrm{Span}\{XA^*Y - YA^*X \mid X, Y \in M\} = \{XA^* - A^*X \mid X \in M\}$$

pf For $0 \le j \le d$

$$A^* E_j = E_{j-1} A^* E_j + E_j A^* E_j + E_{j+1} A^* E_j \qquad (1)$$

$$E_j A^* = E_j A^* E_{j-1} + E_j A^* E_j + E_j A^* E_{j+1} \qquad (2)$$

where $E_{-1} = 0$, $E_{d+1} = 0$

For $0 \le i \le d$

Sum (1) over $j = 0, 1, \ldots, i$

Sum (2) over $j = 0, 1, \ldots, i$

Take difference of these sums

Get

$$E_i A^* E_{i+1} - E_{i+1} A^* E_i = L_i A^* - A^* L_i$$

$$L_i = E_0 + \cdots + E_i$$

Obs

$$M = \mathrm{Span}\{E_i\}_{i=0}^{d} = \mathrm{Span}\{L_i\}_{i=0}^{d}$$

Now

$$\mathrm{Span}\{ XA^*Y - YA^*X \mid X, Y \in M \}$$

$$= \mathrm{Span}\{ E_i A^* E_j - E_j A^* E_i \mid 0 \le i, j \le d \}$$

$$= \mathrm{Span}\{ E_i A^* E_{i+1} - E_{i+1} A^* E_i \mid 0 \le i \le d \}$$

$$= \mathrm{Span}\{ L_i A^* - A^* L_i \mid 0 \le i \le d \}$$

$$= \{ XA^* - A^*X \mid X \in M \}$$

$\square$

LEM 88 Assume

$$E_i A^* E_j = \begin{cases} 0 & \text{if } |i-j|>1 \\ \neq 0 & \text{if } |i-j|=1 \end{cases} \qquad 0 \le i,j \le d$$

Given $\beta, \gamma, \delta \in \mathbb{F}$. TFAE

(i) $\{\theta_i\}_{i=0}^{d}$ is (β,γ,δ)-rec.

(ii) A commutes with

$$A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*$$

pf Similar to the pf of L 23. $\square$

thm 89 Assume A, A^* is a TD pair on V. Then $\exists$ a sequence of scalars $\beta, \gamma, \gamma^*, \delta, \delta^*$ from $\mathbb{F}$ such that

$$0 = [A,\ A^2A^* - \beta AA^*A + A^*A^2 - \gamma(AA^* + A^*A) - \delta A^*], \qquad \text{TD1}$$

$$0 = [A^*,\ A^{*2}A - \beta A^*AA^* + AA^{*2} - \gamma^*(A^*A + AA^*) - \delta^* A] \qquad \text{TD2}$$

$$[r,s] = rs - sr$$

the sequence is unique provided the diameter of A, A^* is ≥ 3.

pf [sim to Prop 35] Let

$$\Phi = (A; \{E_i\}_{i=0}^d; A^*; \{E_i^*\}_{i=0}^d)$$

denote a TD system for A, A^*.

Recall the Φ-split decomp $\{U_i\}_{i=0}^d$ of V, along with maps

$$R,\ L,\quad \{F_i\}_{i=0}^d$$

First assume $d \geq 3$.

By L87 $\exists$ $\{\alpha_i\}_{i=0}^d$ in $\mathbb{F}$ s.t.

$$A^2A^*A - AA^*A^2 = \sum_{i=1}^d \alpha_i (A^iA^* - A^*A^i) \qquad (*)$$

claim 1 $\quad \alpha_i = 0 \quad 4 \le i \le d$

pf cl Suppose not, and set

$$t = \max\{ i \mid 4 \le i \le d, \ \alpha_i \ne 0 \}$$

In (*) mult each term on left by F_t and right by F_0

Get

$$0 = \underbrace{R^t F_0}_{\ne 0} \overbrace{(\theta_0^* - \theta_t^*)}^{\ne 0} \alpha_t^{\ne 0}$$

$\ne 0$ since $R^t : U_0 \to U_t$ is inj

cont. ✓

By cl 1

$$A^2 A^* A - A A^* A^2 = \sum_{i=1}^{3} \left(A^i A^* - A^* A^i \right) \alpha_i \qquad (**)$$

claim 2 $\quad \alpha_3 \ne 0$

pf cl Suppose $\alpha_3 = 0$. In (**) mult each term on left by F_3 and right by F_0

Get

$$\underbrace{R^3 F_0}_{\ne 0} \underbrace{(\theta_1^* - \theta_2^*)}_{\ne 0} = 0$$

cont.

✓

Define

$$C = \alpha_1 A^* + \alpha_2 (AA^* + A^*A) + \alpha_3 (A^2A^* + A^*A^2)$$
$$+ (\alpha_3 - 1) AA^*A$$

Then $AC - CA =$

$$\sum_{i=1}^{3} (A^iA^* - A^*A^i)\alpha_i + AA^*A^2 - A^2A^*A$$

$= 0$

So

$$[A, C] = 0 \qquad (\star)$$

Divide C by α_3

Define

$$\beta = \frac{1}{\alpha_3} - 1, \qquad \gamma = \frac{-\alpha_2}{\alpha_3}, \qquad \delta = \frac{-\alpha_1}{\alpha_3}$$

$(\star)$ becomes TD1

For $2 \le i \le d-1$

In TD1, mult each term on Left by F_{i+1} and right by F. Get

$$0 = \underbrace{R^3 F_{i-2}}_{\overset{\shortparallel}{0}\ \text{L77}} \left(\theta^*_{i-2} - (\beta+1)\theta^*_{i-1} + (\beta+1)\theta^*_i - \theta^*_{i+1} \right)$$

So $\{\theta_i^*\}_{i=0}^d$ is β-rec.

$\exists\ \gamma^* \in \mathbb{F}$ s.t. $\{\theta_i^*\}_{i=0}^d$ is (β, γ^*)-rec

$\exists\ \delta^* \in \mathbb{F}$ .. $(\beta, \gamma^*, \delta^*)$-rec.

Now by L88 $\beta, \gamma^*, \delta^*$ sat TD2.

<u>Uniqueness</u> Let $\beta, \gamma, \gamma^*, \delta, \delta^*$ denote any scalars in $\mathbb{F}$

that sat TD1, TD2

By L88

$\{\theta_i\}_{i=0}^d$ is (β, γ, δ)-rec

$\rightarrow$ -- (β, γ)-rec

$\rightarrow$ β-rec

β, γ, δ uniquely det since $d \geq 3$.

Similarly γ^*, δ^* are unique.

Done for $d \geq 3$.

Case $d \leq 2$ ex. $\square$

Thm 90 Assume A, A^* is TD pair on V with eigenval seq $\{\theta_i\}_{i=0}^d$ and dual eigenval seq $\{\theta^*_i\}_{i=0}^d$.

Let $\beta, \gamma, \gamma^*, \delta, \delta^*$ be as in Thm 89.

(i) $\beta+1$ equals

$$\frac{\theta_{i-2}-\theta_{i+1}}{\theta_{i-1}-\theta_i}, \qquad \frac{\theta^*_{i-2}-\theta^*_{i+1}}{\theta^*_{i-1}-\theta^*_i}$$

for $2 \le i \le d-1$

(ii) $\gamma = \theta_{i-1} - \beta\theta_i + \theta_{i+1}$ $\qquad 1 \le i \le d-1$

(iii) $\gamma^* = \theta^*_{i-1} - \beta\theta^*_i + \theta^*_{i+1}$ $\qquad$ "

(iv) $\delta = \theta_{i-1}^2 - \beta\theta_{i-1}\theta_i + \theta_i^2 - \gamma(\theta_{i-1}+\theta_i)$ $\qquad 1 \le i \le d$

(v) $\delta^* = \theta^{*2}_{i-1} - \beta\theta^*_{i-1}\theta^*_i + \theta^{*2}_i - \gamma^*(\theta^*_{i-1}+\theta^*_i)$ $\qquad$ "

pf From the pf of Thm 89. $\square$