Lecture 22 Wednesday Oct 23

Continue to discuss the split decomp of a

TD system

$\mathbb{F}$ = any field

V = vectorspace$/\mathbb{F}$ with finite pos dim

Fix TD system on V:

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{d})$$

Recall Φ-split decomp $\{U_i\}_{i=0}^{d}$ of V:

$$U_i = (E_0^*V + \cdots + E_i^*V) \cap (E_iV + \cdots + E_dV) \qquad 0 \le i \le d.$$

Recall $\rho_i = \dim U_i$

Next goal: show $\rho_{i-1} \le \rho_i$ $\quad 1 \le i \le d/2$

DEF. 74 Set

$$R = A - \sum_{h=0}^{d} \theta_h F_h$$

$$L = A^* - \sum_{h=0}^{d} \theta_h^* F_h$$

LEM 75 For $0 \le i \le d$ the following hold on U_i:

$$R = A - \theta_i I, \qquad L = A^* - \theta_i^* I$$

pf recall F_h is projection onto U_h for $0 \le h \le d$. $\square$

COR 76 For $0 \le i \le d$

$$R U_i \subseteq U_{i+1},$$

$$L U_i \subseteq U_{i-1}$$

pf By L75 and Lm 67 (ii) $\square$

LEM 77 For $0 \le i \le j \le d$ the

lin trans

$$\begin{aligned} U_i &\longrightarrow U_j \\ v &\to R^{j-i} v \end{aligned}$$

is injective if $i+j \le d$, a bijection if $i+j = d$,

and surjective if $i+j \ge d$.

The lin trans

$$\begin{aligned} U_j &\longrightarrow U_i \\ v &\to L^{j-i} v \end{aligned}$$

is an injection if $i+j \ge d$, a bijection if $i+j = d$,

and surjective if $i+j \le d$.

[Caution: above maps not inverses, even if $i+j = d$]

Pf Consider R:

Case $i+j \le d$:

Given $v \in U_i$ such that $R^{d-i}v = 0$ show $v = 0$.

$$0 = R^{d-i} v$$

$$= (A - \theta_{d} I) \cdots (A - \theta_{i+1} I)(A - \theta_i I) v$$

So

$$v \in E_i V + E_{i+1} V + \cdots + E_d V$$

$$\subseteq E_0 V + \cdots + E_d V$$

Also

$$v \in U_i$$

$$\subseteq U_0 + \cdots + U$$

$$= E_0^* V + \cdots + E_i^* V$$

So

$$v \in (E_0^* V + \cdots + E_i^* V) \cap (E_0 V + \cdots + E_d V)$$

$$= 0 \qquad (\text{by L 66 applied to } \Phi^{\Downarrow})$$

Case $i+j=d$: U_i, U_j have same dim so above

$\ln j$ is big.

Case $i+j \geq d$:

Given $w \in U_j$ find $v \in U_i$ s.t. $R^{j-i} v = w$

Consider map

$$\begin{aligned} U_{d-j} &\longrightarrow U_j \\ u &\longrightarrow R^{d-2j} u \end{aligned}$$

this is a bij.

So $\exists$ $u \in U_{d-j}$ s.t. $R^{d-2j} u = w$

Define $$v = R^{i+j-d} u$$

then $v \in U_i$ and $R^{j-i} v = w$.

$\square$

The pf for L is similar.

LEM 78 $\quad p_{i+1} \le p_i$ for $1 \le i \le d/2$

pf $\quad$ the map

$$U_{i+1} \to U_i$$
$$v \to Rv$$

is an inj by L77 $\quad$ so

$$\underset{\underset{\displaystyle p_{i+1}}{\shortparallel}}{\dim U_{i+1}} \le \underset{\underset{\displaystyle p_i}{\shortparallel}}{\dim U_i} \qquad \square$$

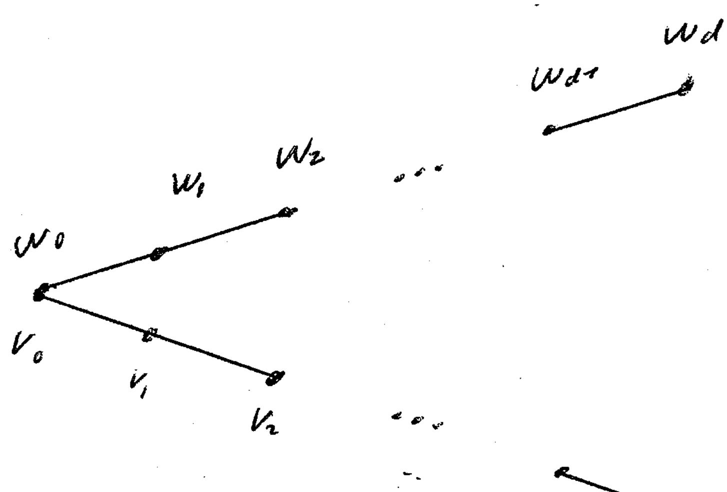
Next goal: the tetrahedron diagram

<u>Notation</u> Given a decomp $\{V_i\}_{i=0}^{d}$ of V

Represent it by a dotted line segment

Given two decomps of V:

$\{V_i\}_{i=0}^{d}$ $\{W_i\}_{i=0}^{d}$



means

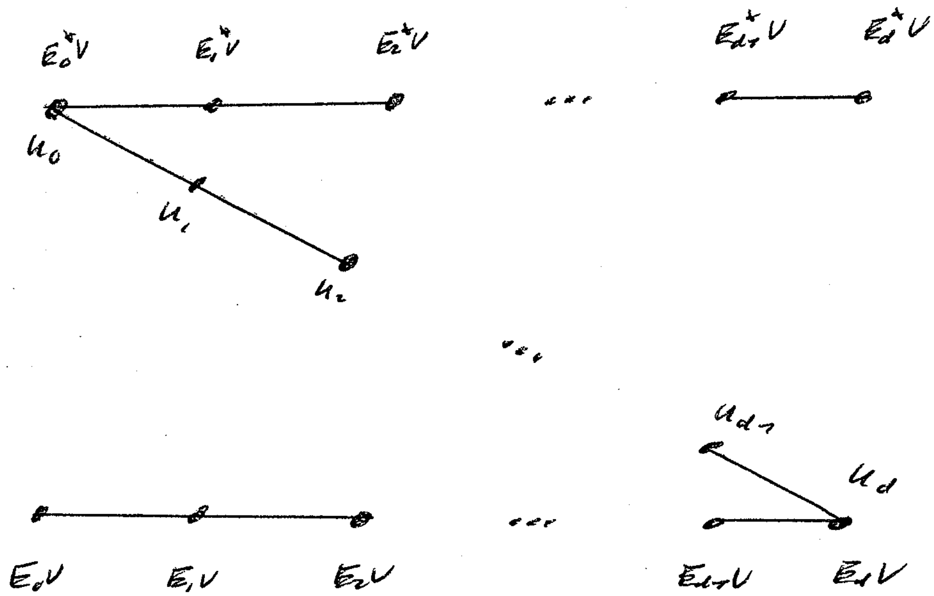
$$\sum_{h=0}^{i} V_h = \sum_{h=0}^{i} W_h \quad \text{for } 0 \le i \le d$$

Recall the Φ-split decomp of V satisfies

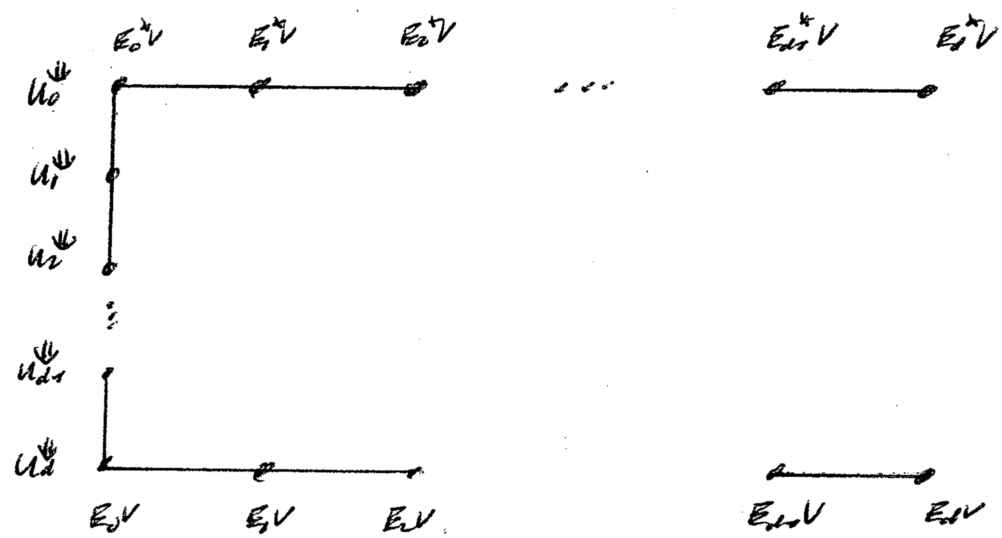
$$U_0+U_1+\cdots+U_i = E_0^*V+\cdots+E_i^*V \qquad 0\le i\le d$$

$$U_i+U_{i+1}+\cdots+U_d = E_iV+\cdots+E_dV$$

Corresp diagram is

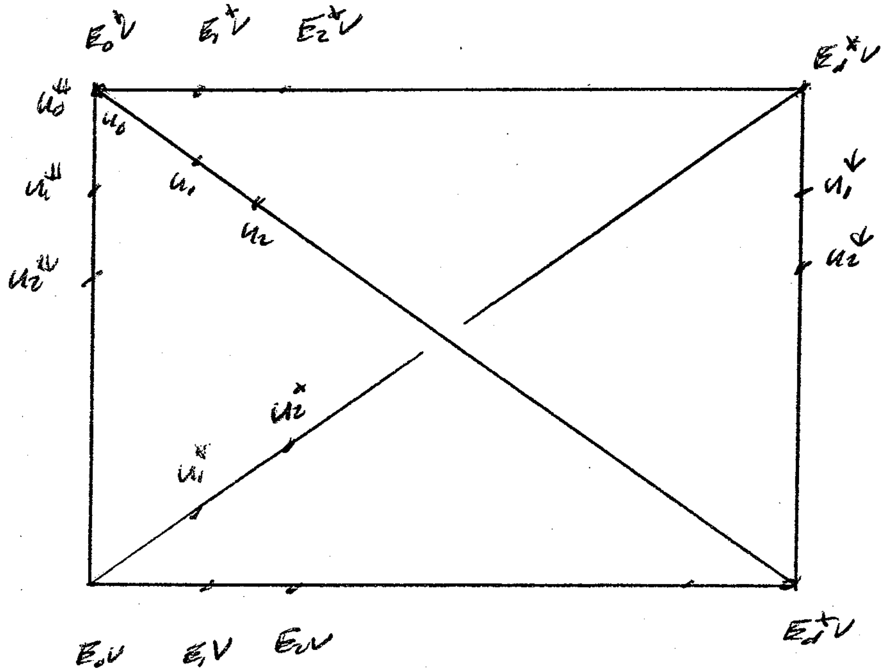


Apply this to $\Phi^{\Downarrow}$ to get



Other relatives of Φ give similar diagrams.

Altogether we get



"tetrahedron diagram for Φ"

Notation Let $\{s_i\}_{i=0}^{d}$ denote

a sequence of pos integers whose sum is dim V.

A flag on V of shape $\{s_i\}_{i=0}^{d}$ is a nested

sequence of subspaces

$$V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d$$

such that

$$\dim V_i = s_0 + s_1 + \cdots + s_i \quad \text{for } 0 \le i \le d.$$

So $V_d = V$.

Example: let $\{W_i\}_{i=0}^{d}$ denote a decomp of V

define $s_i = \dim W_i$ for $0 \le i \le d$.

Define $V_i = W_0 + \cdots + W_i$ $\quad 0 \le i \le d$

then $\{V_i\}_{i=0}^{d}$ is a flag on V of shape $\{s_i\}_{i=0}^{d}$

Given 2 flags on V, denoted

$$\{V_i\}_{i=0}^{d}, \qquad \{V_i'\}_{i=0}^{d}$$

Call these opposite whenever $\exists$ decomp $\{W_i\}_{i=0}^{d}$

of V s.t.

$V_i = W_0 + \cdots + W_i \qquad 0 \le i \le d$

$V_i' = W_d + \cdots + W_{d-i}$

In this case

$$W_i = V_i \cap V_{d-i}' \qquad 0 \le i \le d$$

$$V_i \cap V_j' = 0 \quad \text{if} \quad i+j < d \quad (0 \le i, j \le d)$$

So the decomp $\{W_i\}_{i=0}^d$ is determined by the pair of opp flags. "associated decomp"

DEF 79 For our TD system Φ we now define 4 flags on V, denoted $[0], [D], [0^*], [D^*]$

Each flag has shape $\{\rho_i\}_{i=0}^d$

flag	ith component
$[0]$	$E_0V + \cdots + E_iV$
$[D]$	$E_DV + \cdots + E_{D-i}V$
$[0^*]$	$E_0^*V + \cdots + E_i^*V$
$[D^*]$	$E_D^*V + \cdots + E_{D-i}^*V$

LEM 80 The four flags in Def 79 are mutually opposite.

pf Show $[0^*]$, $[0]$ are opp.

Consider Φ-split decomp $\{U_i\}_{i=0}^d$ of V

For $0 \le i \le d$

$$i\text{th component of } [0^*] = E_0^* V + \cdots + E_i^* V$$
$$= U_0 + \cdots + U_i$$

$$i\text{th component of } [0] = E_0 V + \cdots + E_{d-i} V$$
$$= U_d + \cdots + U_{d-i}$$

Rest of pf is sim. $\square$

Notation Given ordered pair of dist flags in Def 79

$$[\alpha], \quad [\beta]$$

let $[\alpha, \beta]$ denote the associated decomp of V.

We have

decomp	ith subspace of decomp
$[0, D]$	E_iV
$[0^*, D^*]$	E_i^*V
$[0^*, D]$	$(E_0^*V+\cdots+E_i^*V)\cap(E_iV+\cdots+E_DV)$
$[0^*, 0]$	$(E_0^*V+\cdots+E_i^*V)\cap(E_0V+\cdots+E_{D-i}V)$
$[D^*, 0]$	$(E_0^*V+\cdots+E_{D-i}^*V)\cap(E_0V+\cdots+E_{D-i}V)$
$[D^*, D]$	$(E_0^*V+\cdots+E_{D-i}^*V)\cap(E_iV+\cdots+E_DV)$

We now describe the actions of A, A^* on the above decomps.

Thm 81 Let $\{W_i\}_{i=0}^{d}$ denote any one of the above 6 decomps of V_0. Then for $0 \le i \le d$ the action of A, A^* on W_i is:

name	A action	A^* action
$[0, D]$	$(A-\theta_i I) W_i = 0$	$A^* W_i \subseteq W_{i-1} + W_i + W_{i+1}$
$[0^*, D^*]$	$A W_i \subseteq W_{i-1} + W_i + W_{i+1}$	$(A^* - \theta_i^* I) W_i = 0$
$[0^*, D]$	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i-1}$
$[0^*, 0]$	$(A - \theta_{d-i} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_i^* I) W_i \subseteq W_{i-1}$
$[D^*, 0]$	$(A - \theta_{d-i} I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{d-i}^* I) W_i \subseteq W_{i+1}$
$[D^*, D]$	$(A - \theta_i I) W_i \subseteq W_{i+1}$	$(A^* - \theta_{d-i}^* I) W_i \subseteq W_{i+1}$

pf Rows $[0, D]$ and $[0^*, D^*]$ are from def of TD system.
Row $[0^*, D]$ is from Thm 67 (ii).
To get remaining rows, apply thm to relatives of Φ. $\square$