LECTURE 21 MONDAY OCT 21

We continue to discuss the split decomposition of a TD system.

Recall:

$\mathbb{F}$ = any field

V = vector space/$\mathbb{F}$ with finite pos dimension

Fix a TD system on V:

$$\Phi = (A; \{E_i\}_{i=0}^{d}; A^*; \{E_i^*\}_{i=0}^{\delta})$$

For integers i, j define

$$V_{ij} = (E_0^*V + \cdots + E_i^*V) \cap (E_jV + \cdots + E_dV)$$

LEM 66 We have $d=\delta$. Moreover

$$V_{ij} = 0 \quad \text{if} \quad i<j \quad (0 \le i,j \le d) \qquad *$$

pf Switching A, A^* if nec, WLOG $\delta \le d$.

First show $(*)$. To do this, show

$$V_{0r} + V_{1,r+1} + \cdots + V_{d-r,d} \qquad **$$

is 0 for $0 < r \le d$

Let r be given. Let W denote the sum $**$.

By L65 (iii), (iv)

$$AW \subseteq W, \qquad A^*W \subseteq W$$

So $W = 0$ or $W = V$ by the irred of V.

Show $W=0$: By Def 63 each term of $**$ is contained in

$$E_rV + \cdots + E_dV$$

So

$$W \subseteq E_rV + \cdots + E_dV$$

We assume $r>0$ so $W \subsetneq V$. So $W=0$

We have shown $(**)$ is 0 for $0<r\le d$, so $(*)$ holds.

show $d = \delta$ Suppose $d \neq \delta$ so $\delta < d$

Set $i = \delta$, $j = d$ in $(*)$ to get

$$V_{\delta d} = 0$$

By L 64

$$V_{\delta d} = E_d V$$

Cont. So $d = \delta$. $\square$

thm 67 For any subspaces $\{U_i\}_{i=0}^d$ of V

TFAE

(i) $U_i = (E_0^* V + \cdots + E_i^* V) \cap (E_i V + \cdots + E_d V)$ $\quad 0 \le i \le d$

(ii) $\{U_i\}_{i=0}^d$ is a decomposition of V, and both

$(A - \theta_i I) U_i \subseteq U_{i+1}$, $\quad (A^* - \theta_i^* I) U_i \subseteq U_{i-1}$

for $0 \le i \le d$

(iii) For $0 \le i \le d$, both

$$U_i + U_{i+1} + \cdots + U_d = E_i V + \cdots + E_d V, \qquad (1)$$

$$U_0 + U_1 + \cdots + U_i = E_0^* V + \cdots + E_i^* V. \qquad (2)$$

pf (i) → (ii) To get the inclusions set $i = j$ in L65

and note $U_i = V_{ii}$.

Show $\{U_i\}_{i=0}^d$ is a decomp of V:

Show

$$V = \sum_{i=0}^d U_i$$

Define $W = \sum_{i=0}^{d} U_i$

By the inclusions, $AW \subseteq W$ and $A^*W \subseteq W$.

So $W = 0$ or $W = V$. Also W contains

$U_0 = E_0^* V \neq 0$. So $W = V$.

Show the sum $V = \sum_{i=0}^{d} U_i$ is direct.

Suf to show

$$U_i \cap (U_0 + \cdots + U_{i-1}) = 0 \qquad 1 \le i \le d$$

Let i be given. For $0 \le j \le i-1$,

$$U_j \subseteq E_0^*V + \cdots + E_j^*V \subseteq E_0^*V + \cdots + E_{i-1}^*V$$

So

$$U_0 + \cdots + U_{i-1} \subseteq E_0^*V + \cdots + E_{i-1}^*V$$

Also

$$U_i \subseteq E_iV + \cdots + E_dV$$

So

$$U_i \cap (U_0 + \cdots + U_{i-1}) \subseteq (E_iV + \cdots + E_dV) \cap (E_0^*V + \cdots + E_{i-1}^*V)$$

$$= V_{i-1,i}$$

$$= 0 \qquad \text{by L66}$$

Show $U_i \neq 0 \quad 0 \le i \le d$:

We have $\quad U_0 = E_0^* V \neq 0 \qquad U_d = E_d V \neq 0$

Suppose $\exists\, i \; (1 \le i \le d-1)$ s.t. $U_i = 0$

Then $\quad U_0 + U_1 + \cdots + U_{i-1}$ is non 0 proper subspace of V that is invariant under A, A^*, contradicting the irred of V.

We have shown that $\{U_i\}_{i=0}^{d}$ is a decomp of V.

$(ii) \to (iii)$ $\qquad$ Show (i)

Define

$$W = U_i + \cdots + U_d$$
$$Z = E_i V + \cdots + E_d V$$

show $Z \subseteq W$: $\qquad$ Define

$$X = \prod_{h=0}^{i-1} (A - \theta_h I)$$

obs

$$Z = XV$$

Also using the inclusions in (ii)

$$XV \subseteq W$$

So $Z \subseteq W$

show $W \subseteq Z$: Define

$$Y = \prod_{h=i}^{d} (A - \theta_h I)$$

obs

$$Z = \{ v \in V \mid Yv = 0 \}$$

By the inclusions in (ii)

$$Y U_j = 0 \qquad \text{for } i \le j \le d$$

so $YW = 0$

So $W \subseteq Z$

We have shown (1). (2) is sim.

(iii) $\to$ (i) First show the sum $U_0 + \cdots + U_d$ is direct. To this end, show

$$(U_0 + \cdots + U_{i-1}) \cap U_i = 0 \qquad 1 \le i \le d$$

Let i be given

Obs

$$(U_0+\cdots+U_{i-1}) \cap U_i \subseteq (E_0^*V+\cdots+E_{i-1}^*V) \cap (E_iV+\cdots+E_dV)$$
$$= V_{i-1,i}$$
$$= 0$$

So $U_0+\cdots+U_d$ is direct.

Now for $0 \le i \le d$,

$$U_i = (U_0+\cdots+U_i) \cap (U_i+\cdots+U_d)$$
$$= (E_0^*V+\cdots+E_i^*V) \cap (E_iV+\cdots+E_dV)$$

$\square$

DEF 68 By the <u>Φ-split decomposition of V</u> we mean the sequence $\{U_i\}_{i=0}^d$ that satisfies the conditions of Th 67.

Next goal:

For the split decomp $\{U_i\}_{i=0}^d$ show

- E_i^*V, U_i, E_iV have same dim $(=\rho_i)$
- $\rho_i = \rho_{d-i}$ for $0 \le i \le d$
- $\rho_{i-1} \le \rho_i$ for $1 \le i \le d/2$

DEF 69 For $0 \le i \le d$ define $F_i \in \mathrm{End}(V)$

by $$(F_i - I)U_i = 0$$

$$F_i U_j = 0 \quad \text{if } i \ne j \qquad (0 \le j \le d)$$

So F_i is the projection onto U_i.

Define $F_{-1} = 0$, $F_{d+1} = 0$.

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Obs

$$F_i F_j = \delta_{ij} F_i \qquad 0 \le i, j \le d$$

$$I = \sum_{i=0}^d F_i$$

$$U_i = F_i V \qquad 0 \le i \le d$$

LEM 70 For $0 \le i < j \le d$,

(i) $E_i F_j = 0$

(ii) $F_i E_j = 0$

(iii) $E_j^* F_i = 0$

(iv) $F_j E_i^* = 0$

pf (i)

$$
\begin{aligned}
E_i F_j V &= E_i U_j \\
&\subseteq E_i (U_j + \cdots + U_d) \\
&= E_i (E_j V + \cdots + E_d V) \\
&= 0
\end{aligned}
$$

(ii)

$$
\begin{aligned}
F_i E_j V &\subseteq F_i (E_j V + \cdots + E_d V) \\
&= F_i (U_j + \cdots + U_d) \\
&= 0
\end{aligned}
$$

(iii), (iv) sim. $\square$

LEM 71 For $0 \le i \le d$,

(i) $F_i E_i F_i = F_i$

(ii) $E_i F_i E_i = E_i$

(iii) $F_i E_i^* F_i = F_i$

(iv) $E_i^* F_i E_i^* = E_i^*$

pf (i)

$$F_i = F_i^2$$
$$= F_i(E_0 + \cdots + E_d) F_i$$

[For $0 \le j \le d$ $F_i E_j = 0$ if $j > i$ and $E_j F_i = 0$ if $j < i$]

$$= F_i E_i F_i$$

(ii)–(iv) Sim. $\square$

LEM 72 For $0 \le i \le d$,

(i) the linear trans

$$\begin{aligned} U_i &\to E_i V \\ v &\to E_i v \end{aligned} \qquad\qquad \begin{aligned} E_i V &\to U_i \\ v &\to F_i v \end{aligned}$$

are bijections, and moreover they are inverses.

(ii) the linear trans

$$\begin{aligned} U_i &\to E_i^* V \\ v &\to E_i^* v \end{aligned} \qquad\qquad \begin{aligned} E_i^* V &\to U_i \\ v &\to F_i v \end{aligned}$$

are bijections. Moreover they are inverses.

pf (i) they are inverses by L71 (i), (ii).

So they are bijections.

(ii) Sim. $\square$

Cor 73 For $0 \le i \le d$ the dimensions of

$$E_i V, \quad U_i, \quad E_i^* V$$

are equal. Denoting this dim by ρ_i we have

$$\rho_i = \rho_{d-i}$$

pf By L72 the dimensions of

$$E_i V, \ U_i, \quad E_i^* V$$

are equal. Call it ρ_i. Show $\rho_i = \rho_{d-i}$

Suf to show

$$\dim E_i^* V = \dim E_{d-i}^* V \qquad (*)$$

We just showed

$$\dim E_i V = \dim E_i^* V$$

Apply this to $\Phi^{\Downarrow}$ to get

$$\dim E_i V = \dim E_{d-i}^* V$$

This yields $(*)$. $\square$